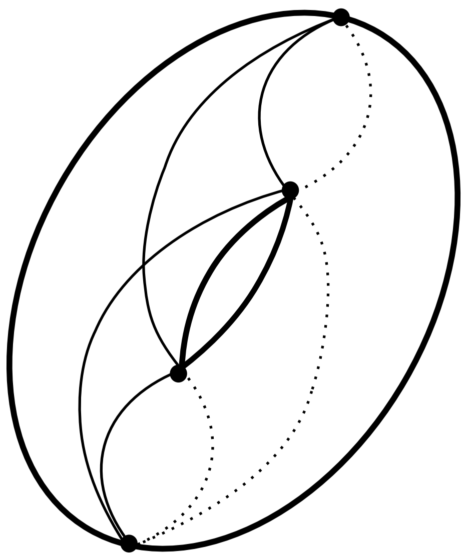
ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Axiomatic Morse Homology

Bachelor Thesis

Gilles Englebert

Wednesday 26th September, 2018

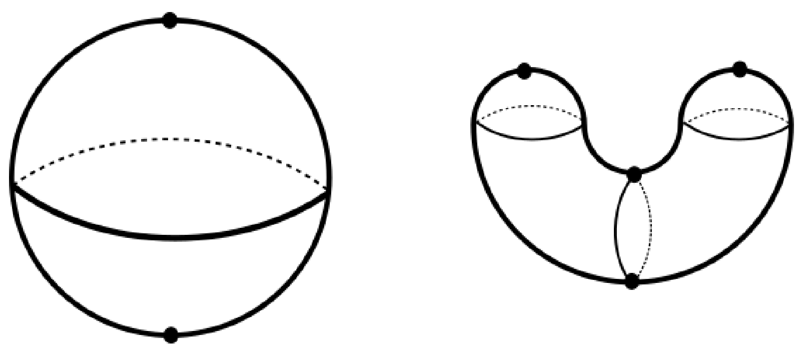


Advisor: Prof. W. Merry

Department of Mathematics, ETH Zürich

Prelude

Morse theory studies the topology of a smooth manifold via the smooth functions that live on it. More precisely, it considers the subset of $C^\infty(M, \mathbb{R})$ given by the Morse functions, which can be thought of as especially well behaved height functions on the manifold. The main idea is that the type, number and position of critical points of such functions can be used to recover topological information about the underlying space. Consider for instance the special case of a sphere in two dimensions:



Two choices of Morse functions on S^2, with critical points marked

Let us assign each critical point p a number according to its type, called its **index**:

$$\mu(p) = \begin{cases} 2, & \text{if } p \text{ is a local maximum} \\ 1, & \text{if } p \text{ is a saddle point} \\ 0, & \text{if } p \text{ is a local minimum} \end{cases}$$

Then we can consider the sum:

$$\eta(S^2) = \sum_{p \text{ critical point}} (-1)^{\mu(p)}$$

Morse functions have the property that on compact manifolds they only admit finitely many critical points, so that the above sum is well defined for Morse functions on S^2.

By calculating the number $\eta(S^2)$ explicitly for both of the pictured cases, we find that it is always equal to 2. This turns out not to be a coincidence, since no matter which Morse function we choose on S^2, this number remains the same. If we consider the torus $\mathbb{T}^2$, i.e. the surface of a donut, and a Morse function on it, such as the one on the front cover, we see that

$$\eta(\mathbb{T}^2) = 0$$

Again, changing the Morse function will not change $\eta(\mathbb{T}^2)$. Now one can play this game also for genus g surfaces $\mathbb{T}_g$ and see that:

$$\eta(\mathbb{T}_g) = 2 - 2g$$

Hence, by the classification of 2-manifolds (which interestingly can also be proven using Morse theory), we see that for any Riemann surface M, we have:

$$\eta(M) = \chi(M)$$

where $\chi(M)$ is the Euler characteristic of M. This is again no coincidence! Let us rewrite the formula for η:

$$\eta(M) = \sum_{k\geq 0}(-1)^k C_k, \qquad C_k = |\{p : p \text{ critical point of index } k\}|$$

This compares nicely to a formula from algebraic topology. Let $M_0 \subset M_1 \subset M_2 = M$ be a cell complex structure on M. Then we have:

$$\chi(M) = \sum_{k\geq 0}(-1)^k N_k, \qquad N_k = \text{Number of } k\text{-cells}$$

It turns out that under certain conditions a Morse function will give rise to a cell complex structure with exactly C_k cells of dimension k. If we for assume this result for a moment, we immediately get:

$$\chi(M) = \sum_{k\geq 0}(-1)^k N_k = \sum_{k\geq 0}(-1)^k C_k = \eta(M)$$

for manifolds which admit these sufficiently regular Morse functions. This means that we can express **topological invariants** such as the Euler characteristic solely **in terms of critical points** of height functions. In fact, a single Morse function, by giving rise to a cell complex structure, suffices in principle to classify manifolds up to homeomorphism type. What we will look at is the so called **Morse homology** of a manifold, which is just the homology of a cell complex obtained by a Morse function, at least from the classical point of view. But that is not how we are going to define it, as we will take a more abstract modern route. We will look at the so called **Morse chain complex** $\mathrm{CM}_\bullet(M; f)$ associated to a manifold with a Morse function f on it. The k-th chain groups will be free $\mathbb{Z}_2$-modules generated by the critical points of index k, and the boundary operator ∂ will be defined by counting solutions to a differential equation. Both approaches, classical and modern, lead to the same result, however modern Morse homology has several advantages over classical Morse homology. One of them is that it lends itself better to our axiomatic treatment, as we will be able to prove by purely analytic arguments that Morse homology is independent of the choice of Morse function. This proof in particular is made cleaner by our choice of route. The independence on the Morse function will then be used afterwards to be able to work with the most appropriate Morse function for a given situation, hence simplifying all further proofs. Another advantage is that the same abstract methods can be applied in situations where there is no cell complex, for instance when the manifold is infinite-dimensional. One important example of this is **Floer homology**, where in some versions, the manifold becomes a loop space and the Morse function gets replaced by the energy functional. Critical points of this functional are then geodesics, which leads to interesting results. In this thesis however we will only ever consider the finite dimensional case. Let us now take a quick look at our method.

The first chapter will focus on defining the Morse homology groups, and in particular showing that they are well defined. We will extend our notion of index to general manifolds, and make concrete the expression ”counting solutions to equations” with the so called **trajectory spaces**.

Then the second chapter will deal with extending the groups to functors and show that the so called **Eilenberg-Steenrod-axioms** are satisfied. We will then present a uniqueness theorem, which states that any two homology theories satisfying the ES-axioms are isomorphic. Since singular homology also satisfies the ES-axioms, we get that Morse homology is in fact isomorphic to singular homology.

Finally, in the last chapter we will explore a number of theorems from algebraic topology in the spirit of Morse homology. We will cover Morse cohomology and Poincaré duality, which turns out is almost trivial to prove in Morse homology. To conclude, we will give a quick proof of the existence of an Eilenberg-Zilber chain map in by exploiting an additivity property of Morse functions.

I would like to express my sincere thanks to Professor Will Merry for introducing me to the world of algebraic topology as well as his excellent guidance throughout the preparation of this thesis.

Contents

Chapter 1

Morse Homology

Morse Homology is family of functors

$$\mathrm{HM}_k : \mathbf{cMan} \to \mathbf{Ab}$$

from the category of closed manifolds into the category of abelian groups. Essentially they exploit that the topology of a manifold restricts which kinds of Morse functions it can have. As topological invariants they are similar, but more powerful than for instance the Euler characteristic $\chi(M)$ or the number of connected components of the manifold $\#\pi_0(M)$. The goal of this chapter is to construct the absolute Morse homology groups $\mathrm{HM}_k(M;\mathbb{Z}_2)$ with $\mathbb{Z}_2$-coefficients, and show that they are well defined. To this end, we will first introduce Morse functions, whose properties are the cornerstones on which the rest of the thesis grounds. After a technical interlude about trajectory spaces, which are instrumental in constructing the so called Morse chain complex in dependence of a Morse function on M, we will show that this dependence only affects the chain level, and can be removed in homology by a diagonal argument. This will then give us the Morse homology groups $\mathrm{HM}_k(M;\mathbb{Z}_2)$. Notice that from now on we will suppress the coefficient group from the notation, since we will never work with different coefficients, and may sometimes write HM_* to denote the Morse homology in some fixed, but arbitrary degree.

1.1 Fundamentals of Morse Theory

Morse Theory is the study of the so called Morse functions, which intuitively can be thought of as height functions with discrete extremal and saddle points.

Definition 1.1 *Let (M,g) be a smooth Riemannian manifold and $f : M \to \mathbb{R}$ a smooth function.*

- *A point $p \in M$ is called a* ***critical point*** *of f if $df_p : T_pM \to \mathbb{R}$ vanishes. The set of all critical points of f will be denoted by* $\mathrm{Crit}(f)$.

- *The **Hessian** of a function f at the point p is the bilinear form*

$$H_p(f) : T_pM \times T_pM \to \mathbb{R}$$

given by the covariant derivative of df_p, i.e.

$$H_p(f)(v,w) = \nabla_V(df_p)(W) := L_V(df_p(W)) - df_p(\nabla_V W)$$

for vector fields V and W on an open neighbourhood of p extending tangent vectors v and w respectively, and $L_X(f)$ is the Lie-derivative of a function f with respect to a vector field X. If $\varphi = (x_1, ..., x_m) : U \subset M \to \Omega$ is a coordinate chart, consider $H_p(f)$ in the basis $\frac{\partial}{\partial x_i}$. It is given by

$$H_p(f)\left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}\right) = \frac{\partial^2(f \circ \varphi^-1)}{\partial x_i \partial x_j}(\varphi(p)) =: (M_p(f))_{ij}$$

It is a straightforward exercise to show that for p a critical point of f, $H_p(f)$ does not depend on the choice of connection and is symmetric.

Definition 1.2 *Let $f : M \to \mathbb{R}$ be a smooth function.*

- *A critical point $p \in M$ of f is said to be **non-degenerate** if $M_p(f)$ is invertible.*
- *A function $f : M \to \mathbb{R}$ is called a **Morse function** if none of its critical points are degenerate.*
- *The **Morse index** of a critical point $p \in M$ of a Morse function $f : M \to \mathbb{R}$, denoted by $\mu(p)$ is the dimension of the maximal subspace of T_pM on which $H_p(f)$ is negative definite. The set of all critical points of index $k \in \mathbb{Z}$ will be denoted by* $\mathrm{Crit}_k(f)$.

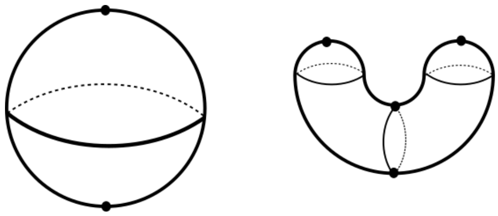
From linear algebra we know that real symmetric matrices are diagonalizable, hence in particular $M_p(f)$ is diagonalizable. In fact $\mu(p)$ is the number of negative eigenvalues of $M_p(f)$, which is independent of the choice of the coordinate chart. If the manifold M is thought of as a mountain where the point p has height $f(p)$, then $\mu(p)$ describes the number of directions in which an athlete could ski downhill.

Example 1.3 *Let $S^n = \{(x_0, ..., x_n) : \sum_{i=0}^n x_i^2 = 1\} \subset \mathbb{R}^{n+1}$ denote the unit n-sphere with $n > 0$. Define*

$$f_n : S^n \to \mathbb{R} \tag{1.1}$$

$$(x_0, ..., x_n) \mapsto x_n \tag{1.2}$$

i.e. f_n is the projection onto the last coordinate. This defines a Morse function on S^n, with exactly two critical points, namely the south pole $S = (0, \ldots, 0, -1)$ and the north pole $N = (0, \ldots, 0, 1)$. As N is a local maximum, its index has to be $\mu(N) = n$, for one can ski downhill in every direction. The south pole on the other hand must



Examples of Morse functions on S^2: f_2 and h

have index $\mu(S) = 0$*, since it's a local minimum, and hence there is no direction in which one could ski further downhill. For* $n = 2$*, one can also imagine the so called "hot dog"-Morse function h. We will leave it to the reader to find out why it is called that way.*

Morse functions behave nicely under product operations.

Proposition 1.4 *Let* $f : M \to \mathbb{R}$ *and* $g : N \to \mathbb{R}$ *be two Morse functions. Then*

$$\begin{aligned} f \oplus g : M \times N &\to \mathbb{R} \\ (p,q) &\mapsto f(p) + g(q) \end{aligned}$$

is also a Morse function and $\mathrm{Crit}(f \oplus g) = \mathrm{Crit}(f) \times \mathrm{Crit}(g)$*, where* μ *behaves additively, i.e.:*

$$\mu(p,q) = \mu(p) + \mu(q)$$

Proof The set of critical points of $f \oplus g$ is given by $\mathrm{Crit}(f \oplus g) = \mathrm{Crit}(f) \times \mathrm{Crit}(g)$. This is because given $\hat{p} \in T_pM$ and $\hat{q} \in T_qN$ we have

$$d(f \oplus g)_{(p,q)}(\hat{p}, \hat{q}) = df_p(\hat{p}) + dg_q(\hat{q}).$$

So (p,q) is critical exactly when p and q respectively are. For each critical point $(p,q) \in \mathrm{Crit}(f \oplus g)$ we know that $M_p(f)$ and $M_q(g)$ are non-degenerate for given charts φ and ψ respectively, hence also :

$$M_{(p,q)}(f \oplus g) = \begin{bmatrix} M_p(f) & 0 \\ 0 & M_q(g) \end{bmatrix}$$

if we consider the product chart $\varphi \times \psi$ to express the hessian of $f \oplus g$ in local coordinates. From this it is clear that μ behaves additively, since it just counts the number of negativ eigenvalues of the matrices representing the hessian. □

Example 1.5 *Define the n-Torus as* $\mathbb{T}^n = (S^1)^n$ *and consider:*

$$\begin{aligned} f : \mathbb{T}^n &\to \mathbb{R} \\ (x_0^0, x_0^1, ..., x_n^0, x_n^1) &\mapsto \sum_{i=0}^{n} x_i^0 \end{aligned}$$

This defines a Morse function, by applying the previous lemma $n - 1$ *times.*

There are a lot of reasons why Morse functions are interesting to work with. One of them is that their critical points are nicely behaved.

Proposition 1.6 *Let $f : M \to \mathbb{R}$ be a Morse function and $p \in M$ a critical point of f. Then there is an open neighborhood $p \in U \subset M$ such that p is the only critical point in U.*

Proof Let $\varphi : U \subset M \to \mathbb{R}^m$ be a coordinate chart on a neighborhood of p with $\varphi(p) = 0$ and set $\Omega := \varphi(U)$. Define the smooth map $g : \Omega \subset \mathbb{R}^m \to \mathbb{R}^m$ by $p \mapsto (\partial_1(f \circ \varphi^{-1})(p), \cdots, \partial_m(f \circ \varphi^{-1})(p))$. Now as p is a critical point of f, $g(0) = 0$. One readily calculates that:

$$dg_x = M_x(f)$$

so that the non-degeneracy of p implies that dg_0 invertible. Now by the inverse function theorem there is a neighborhood $V \subset \Omega$ of 0, such that $g|_V$ is a diffeomorphism and in particular injective. But this in turn implies that the open neighborhood $\varphi^{-1}(V)$ of p only contains the critical point p. □

Remark 1.7 *This allows us to conclude that a Morse function on a compact manifold only has finitely many critical points, and in particular only finitely many critical points of a given index, which will be useful later on when we show that the homology of a compact manifold is finitely generated.*

The prudent reader will be glad to hear that Morse functions do in fact exist on every manifold as explained by the following proposition:

Proposition 1.8 *Given any manifold M, not necessarily compact, there is a Morse function $f : M \to \mathbb{R}$.*

A proof of this result can be found in [1].

We will conclude the section by stating and explaining two properties of Morse functions which are essential in the construction of Morse Homology, namely stability and genericity.

Theorem 1.9

- *Morse functions are* ***generic*** *in $(C^\infty(M), \|.\|_{C^\infty(M)})$. Here we mean generic in the sense of Baire, that there is a countable family of open and dense sets $U_i \subset C^\infty(M) (i \in \mathbb{N})$ such that the set $\mathbb{M}$ of Morse functions on M satisfies $\cap_{i \in \mathbb{N}} U_i \subset \mathbb{M}$.*
- *Morse functions are a* ***stable*** *class of functions. Let $f_0, f_1 : M \to \mathbb{R}$ be smooth and let f_0 be Morse. If f_t is a smooth homotopy from f_0 to f_1 then there exists an $0 < \varepsilon \leq 1$ such that for all $t < \varepsilon$ we have that f_t is Morse.*

Proof (Sketch) One can prove the following for a smooth function $f : M \to \mathbb{R}$:

$$f \text{ is Morse} \Leftrightarrow df \pitchfork 0_{TM^\star}$$

where $0_{TM^\star}$ is the zero section in $TM^\star$. Now it is a fact from transversality theory that being transversal to a submanifold is a stable property. For genericity, one embeds M into some $\mathbb{R}^k$, and considers the functions :

$$f - L : M \to \mathbb{R}$$

where $L : \mathbb{R}^k \to \mathbb{R}$ is a linear function. Then one shows that $f - L$ is Morse if and only if $q \in R^k$ is a regular value of an auxiliary smooth function. Now by Sard's theorem, this last assertion is generically satisfied. In particular this also proves the existence of a Morse function on M. $\square$

Remark 1.10 *In the physical world, measurements are never accurate and this has to be taken into account. If we interpret measuring some state of a manifold M as getting a function $f : M \to \mathbb{R}$, we can interpret these two conditions in terms of measurement errors. So for instance stability of Morse functions translates into the fact that they can be observed, even if the measurement device is not absolutely accurate. On the other hand, genericity means that they are the only kind of function measurable.*

1.2 Trajectory Spaces

Let (M, g) now be a closed Riemmanian manifold. We can use it to induce an isomorphism

$$\tilde{g} : Vect(M) \to \Omega^1(M)$$

from the space of tangent vector fields on M to the differential 1-forms on M defined as follows: if $p \in M$, then T_pM is a finite-dimensional vector space together with an inner product $g_p : T_pM \times T_pM \to \mathbb{R}$. Hence there is an isomorphism $\tilde{g}_p : T_pM \to T_pM^\star$ given by $v \mapsto g_p(v, \cdot)$. These maps on individual tangent spaces can be regrouped to give us a map $\bar{g} : TM \mapsto TM^\star$ from the tangent bundle to the cotangent bundle. The metric g on M is smooth, hence $\bar{g}$ can also be shown to be smooth. Finally notice that vector fields are smooth sections of the tangent bundle, whereas differential forms are smooth sections of the cotangent bundle.

We are now able to define:

Definition 1.11 *Let $f : M \to \mathbb{R}$ be smooth. The* ***gradient vector field of*** *f with respect to the metric g is the vector field $\nabla f := (\tilde{g}^{-1})(df)$. In other words it is the unique vector field which satisfies: $g_p(\nabla f(p), V(p)) = df(p)V(p) \forall V \in Vect(M)$.*

Now that we constructed a vector field on M we can consider its flow.

Definition 1.12 *Solutions $\gamma : \mathbb{R} \to M$ to the equation $\dot{\gamma}(t) = -\nabla f(\gamma(t))$ are called* ***gradient flow lines****. For $x, y \in \mathrm{Crit}(f)$ the* ***trajectory space between*** *x* ***and*** *y will be :*

$$M^f_{x,y} = \{\gamma : \dot{\gamma}(t) = \nabla f(\gamma(t)), \lim_{t\to-\infty} \gamma(t) = x, \lim_{t\to+\infty} \gamma(t) = y\}$$

Remark 1.13 *Solutions to the equation exist at least locally from standard ODE Theory, and in fact in this case each solution exists for all time, since M is a compact manifold.*

Intuitively, the negative gradient always points into the direction in which f decreases as quickly as possible. Hence we would expect the following:

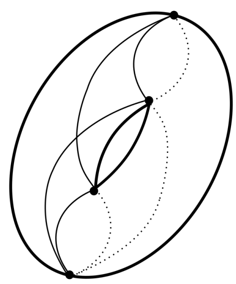
Proposition 1.14 *Let $f : M \to \mathbb{R}$ be smooth. Then f decreases along flow lines.*

Proof

$$\frac{d}{dt} f(\gamma(t)) = df_{\gamma(t)} \dot{\gamma}(t) = g_p(\nabla f(\gamma(t)), \dot{\gamma}(t)) = -\|\dot{\gamma}(t)\|_g^2 < 0$$

□

Remark 1.15 *Let us consider the height function on a slightly tilted torus. We wil explain later, when we introduce the Morse-Smale condition, why the tilt is important. For now, just notice that when plotting the flow lines between any two critical points of consecutive index, there seem to be exactly two paths leading from one to the other. Also for critical points of the same index, there are no paths joining them. Keep this picture in mind for the subsequent section.*



Flow lines between consecutive critical points on the 2-Torus

From now on, the analysis will get tough, so we will instead focus on the concepts and leave out most of the actual derivations. With the definitions out of the way, we can come to the main purpose of this section. We would like to count the number of gradient flow lines between two critical points. However, until now $M^f_{x,y}$ is a rather abstract subset of the function space $C^\infty(\mathbb{R}, M)$ which has either no elements, one element, or infinitely many. The first case arises for instance when $f(x) < f(y)$, for we have seen that that f strictly decreases along flow lines. By the same argument one realizes that $f(x) = f(y)$ and the existence of a flow line γ between x and y leads to the conclusion that γ is constantly equal to $x = y$. Finally, if γ is a non-constant flow line such that $f(x) > f(y)$ then every time shifted $\gamma(\cdot + t)$ is again a flow line.

This follows from the fact that the ODE used in the definition of $M^f_{x,y}$ is autonomous and without boundary constraints.

So in principle we have two problems, firstly $M^f_{x,y}$ is amorphous and secondly even if we knew what $M^f_{x,y}$ looks like we are still not able to count the distinct flow lines in this form. We tackle the first one by taking a look at the classical approach to Morse homology, which is based on:

Definition 1.16

- *The **stable manifold** of a critical point y:*

$$W^s(y) = \{p \in M : \exists \gamma : \mathbb{R} \to M \text{ is a gradient flow line with } \gamma(0) = p \text{ and } \lim_{t\to+\infty} \gamma(t) = y\}$$

- *The **unstable manifold** of a critical point x:*

$$W^u(x) = \{p \in M : \exists \gamma : \mathbb{R} \to M \text{ is a gradient flow line with } \gamma(0) = p \text{ and } \lim_{t\to-\infty} \gamma(t) = x\}$$

Let's take a look at these objects. It turns out it is most useful to look at W^s and W^u through a physics tinted pair of glasses. Suppose there was a point particle at a position $p \in W^s(y)$ at some initial time. If we let it evolve such that at each moment, the momentary velocity of the particle is given by $-\nabla f$, then, sooner or later, it will end up approximately at y. Around y it will become slower and slower, and in the limit, converge to y. In this sense y is an attractive fixed point for all starting positions in $W^s(y)$. In the same vein x is a repulsive fixed point of this evolution for any $p \in W^u(x)$. Furthermore, as their names suggest these two subsets of M are indeed manifolds as is explained by

Theorem 1.17 (Stable- and unstable manifold)

- *The **stable manifold** of a critical point y is a submanifold of M, and diffeomorphic to a $(m - \mu(y))$-cell, for instance in the following manner:*

$$E_{m-\mu(y)} \cong \{v \in T_yM : H_y(v, v) > 0\} \xrightarrow{\cong} W^s(y)$$

- *The **unstable manifold** of a critical point x is a submanifold of M, and diffeomorphic to a $\mu(x)$-cell, for instance in the following manner:*

$$E_{\mu(x)} \cong \{v \in T_xM : H_x(v, v) < 0\} \xrightarrow{\cong} W^u(x)$$

Notice that this makes intuitive sense in the context of the "skier on a mountain" approach to Morse functions. Namely, if the skier rides in one of the directions where the Hessian is negative definite, then he will inevitably go downhill, as a $H_x(v, v)$

means that if he rides in direction v his height, given by f, will decrease. An analogous interpretation holds true for the other case. The great thing about the (un)stable manfolds are that they can be more easily visualized than trajectory spaces. Furthermore as subsets of M they can be intersected, which gives

$$W^u(x) \cap W^s(y) = \{p \in M : \exists \gamma : \mathbb{R} \to M \text{ is a gradient flow line with } \gamma(0) = p, \\ \lim_{t\to-\infty} \gamma(t) = x, \lim_{t\to+\infty} \gamma(t) = y\}$$

Now, this already looks a lot like our trajectory space $M^f_{x,y}$. But in general the intersections of stable and unstable manifolds are not manifolds themselves. This however can be fixed when we assume the following condition on f and g:

Definition 1.18 *(Morse-Smale condition)*
The unstable manifold $W^u(x)$ and the stable manifold $W^s(y)$ intersect transversally for every pair of critical points $x, y \in \mathrm{Crit}(f)$ of f.

There is a purely analytical expression of this condition, which can for instance be found in [3, p.28]. We rely on the more geometric point of view solely to make the concepts clearer. It is an easy consequence of the transversality theorem from differential topology that

Theorem 1.19 *If f and g satisfy the Morse-Smale conditions, then the intersection $W^u(x) \pitchfork W^s(y)$ is itself again a manifold of dimension $\mu(x) - \mu(y)$.*

This condition being tailor made for making sure everything we want to be a manifold is a manifold, it is reassuring to know that it is not very restrictive, for:

Theorem 1.20 *The Morse-Smale condition is generically satisfied, given a Morse function f and a metric g on M. More precisely, given a Morse function f on M the set:*

$$\{\tilde{g} \in \mathfrak{R}(M) : (f, \tilde{g}) \text{ satisfies the Morse-Smale condition}\} \subset \mathfrak{R}(M)$$

is residual, as is for any metric g on M the set:

$$\{\tilde{f} \in C^\infty(M, \mathbb{R}) : (\tilde{f}, g) \text{ is Morse-Smale and } f \text{ is Morse}\} \subset C^\infty(M, \mathbb{R}).$$

Here $\mathfrak{R}(M)$ denotes the subset of $(0, 2)$-tensor fields on M that are Riemannian metrics.

Now that these results have been stated, we will relate them to the trajectory spaces. However, since the $W^u(x) \pitchfork W^s(y)$ is made up of points of M and $M^f_{x,y}$ is a function space, we need to get get points out of the functions. In fact, we can do this by evaluating elements of $M^f_{x,y}$ at 0. To be more concrete:

Theorem 1.21 *The trajectory space $M^f_{x,y}$ can be given a smooth structure in such a way that the following is a diffeomorphism:*

$$\varphi : M^f_{x,y} \to W^u(x) \pitchfork W^s(y)$$
$$\gamma \mapsto \gamma(0)$$

This means that $M^f_{x,y}$ is a smooth manifold of dimension $\mu(x) - \mu(y)$.

Since the ODE defining $M^f_{x,y}$ is autonomous, the space admits a natural symmetry, as gradient flow lines $\gamma(\cdot) \in M^f_{x,y}$ can be shifted in time to get a new curve $\gamma(\cdot + \tau) \in M^f_{x,y}$. This observation is made precise in the following theorem:

Theorem 1.22 *There is an free $\mathbb{R}$-action on $M^f_{x,y}$ which is given by*

$$\mathbb{R} \times M^f_{x,y} \to M^f_{x,y}$$
$$(\tau, \gamma) \to \tau \bullet \gamma$$

where $(\tau \bullet \gamma)(t) := \gamma(t + \tau)$.

The intuition behind this action is that the orbit of a flow line under this action is the set of all flow lines which have the same image in M, i.e. "look the same when drawn". When counting flow lines, we don't want to count ones that look the same twice. With that goal in mind we define the following variant of the trajectory space, where we factor the $\mathbb{R}$-action out:

Definition 1.23 *The* ***unparametrized trajectory space*** *is given by $\hat{M}^f_{x,y} := M^f_{x,y}/\mathbb{R}$.*

For compact manifolds M we can now continue to prove analogous theorems as for the parametrized trajectory space, but, just as a side note, if M is non compact we impose some mild niceness conditions on f, so that we can still guarantee $\hat{M}^f_{x,y}$ to be nice.

Definition 1.24 *The function f is said to be* **coercive***, if $f^{-1}((-\infty, a])$ is compact for every $a \in \mathbb{R}$.*

In this context, a pair (f, g) where f is a Morse function on M and g a Riemannian metric on M, such that (f, g) satisfy the Morse-Smale and f is coercive is called a **nice pair**. If M is compact of course, coercivity is trivially satisfied for every pair, since every sequence in M has a convergent subsequence. However, when showing that Morse Homology satisfies the Eilenberg-Steenrod axioms, we will consider manifolds of the form $\mathbb{R}^k \times M$ with M compact, and there we will need to show niceness explicitly for every (f, g) we use. In general though, Morse functions on non-compact manifolds are not coercive and not easy to deal with, hence we will not work with them except in very specific cases where coercivity can be shown explicitly. We now elaborate on the precise meaning of $\hat{M}^f_{x,y}$ being nice:

Theorem 1.25 *Given a nice pair (f, g), the unparametrized trajectory space $\hat{M}^f_{x,y}$ is a smooth manifold of dimension $\mu(x) - \mu(y) - 1$. This is a consequence of the Morse-Smale condition.*

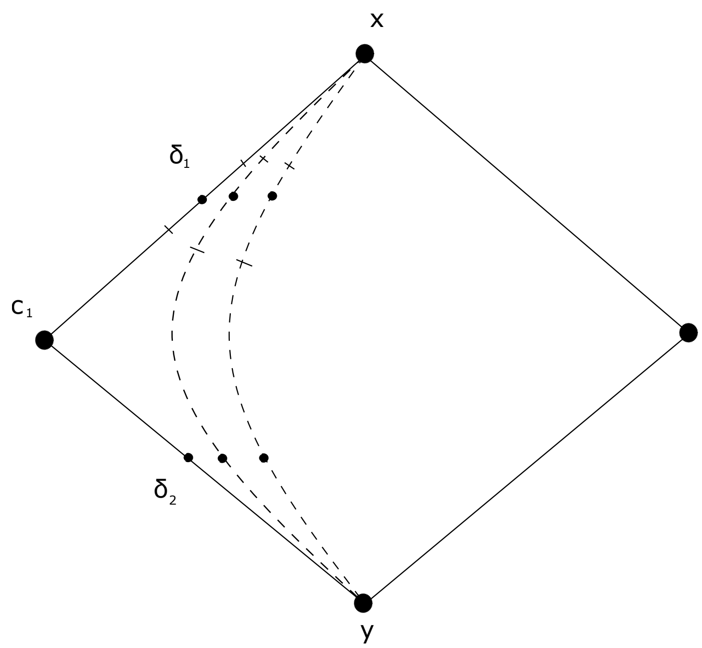
Theorem 1.26 *Given a nice pair (f, g), we have that $\hat{M}^f_{x,y}$ is* ***compact up to broken trajectories of degree*** *$k = \mu(x) - \mu(y)$. This means that for every sequence $(\gamma_n)_{n\in\mathbb{N}} \subset M^f_{x,y}$ we can find a subsequence (which we will not reindex), critical points $x = c_0, ..., c_l = y$, where $l \leq k$, flow lines $\delta_i \in M^f_{c_{i-1},c_i}$ and reparametrisation times $\tau_{i,n} \in \mathbb{R}$ such that*

$$\tau_{i,n} \bullet \gamma_n \xrightarrow{C^\infty_{loc}} \delta_i$$

This is a consequence of coercivity.

Here C^∞_{loc}-convergence of functions $f_i : \mathbb{R} \to M$ to f is defined as $f_i \xrightarrow{C^\infty} f$ on each compact $K \subset \mathbb{R}$.

Example 1.27 *Geometrically, this convergence can be visualized best if $k = 2$.*



The countable family of trajectories γ_n converges to the broken trajectory composed of δ_1 and δ_2. This means explicitly, that if we choose our reparametrisation times $\tau_{1,n}$ in such a way that $(\tau_{1,n} \bullet \gamma_n)(0)$ converges to $\delta_1(0)$ and choose some compact $K \subset \mathbb{R}$ (in the figure it's an interval containing 0), then on this K our family of trajectory $\tau_{1,n} \bullet \gamma_n|_K$ converges to $\delta_1|_K$ in C^∞. By reparametrising γ_n differently, say in such a way that $(\tau_{1,n} \bullet \gamma_n)(0) \to \delta_2(0)$ we obtain trajectories that have the same range, i.e. "look the same when drawn", however now we forced them to converge locally to the lower part of the broken trajectory. This is possible, since the C^∞_{loc}-convergence *only needs C^∞-convergence on each compact set, which means that the asymptotic behaviour of the γ_n is irrelevant.*

Proof The main ingredient of the proof is the theorem of Arzela-Ascoli. Coercivity is needed so that we can get equicontinuity of the trajectories, which we think of as living in $C(M, \mathbb{R})$, and get a converging subsequence. We then prove that the limit also satisfies the ODE characterising trajectory spaces, hence in particular it is smooth and lives in some $M^f_{a,b}$. We then are faced with two possibilities:

- If $(a, b) = (x, y)$, we are done.
- If $(a, b) \neq (x, y)$ we can without loss of generality assume $a \neq x$ and $f(a) < v < f(x)$. This can be shown to imply $\mu(a) < \mu(x)$. We then choose reparametrization times τ_n such that $f((\tau_n \bullet \gamma_n)(0)) = v$. This forces a potential limit of a converging subsequence to converge to a different part of the broken trajectory, as we essentially "push up" the trajectories with this modification. It will turn out in fact that we can make it converge in $M^f_{x,a}$. In some sense, we now managed to break the trajectory once. But this can be repeated at most finitely many times, only stopping when there are no more trajectories to break, since at each breaking, we introduce a new critical point which has an index strictly in between the indices of the current breaking points, and k we start with is finite. □

We have now some very general theorems at our disposal, that we will use to get the more specific results that we nee to construct Morse homology. The first one being

Corollary 1.28 *$\hat{M}^f_{x,y}$ is a compact 0-dimensional manifold, i.e. a finite set, for $\mu(x) = \mu(y) + 1$.*

Proof That it is a manifold of dimension 0 follows from theorem 1.25 using that $\mu(x) - \mu(y) - 1 = 0$. Compactness follows from theorem 1.26, by remembering that in $\hat{M}^f_{x,y}$ reparametrisations of a trajectory represent the same point as that trajectory, and that the action is continuous. □

Finally we are able to count flow lines.

Definition 1.29 *Suppose $\mu(x) = \mu(y) + 1$. Then we say $[x, y] := |\hat{M}^f_{x,y}| \mod 2$ is the number of gradient flow lines (mod 2) between x and y.*

1.3 Morse Complex

Now that the basics are out of the way we can start defining the Morse chain complex, from which we will later construct the corresponding homology.

Definition 1.30 *Suppose $f : M \to \mathbb{R}$ is a Morse function and g is a Riemannian metric on M, such that (f, g) is a nice pair. The **Morse chain complex** $(\mathrm{CM}_\bullet, \partial)_{f,g}$ is given by the following data:*

- *For all $k \in \mathbb{Z}$ let $(\mathrm{CM}_\bullet)_k$ be the free $\mathbb{Z}_2$-module with basis* $\mathrm{Crit}_k(f)$.
- *Let $x \in \mathrm{Crit}_k(f)$. Define then $\partial x := \sum_{y \in \mathrm{Crit}_{k-1}(f)} [x,y] y$ and extend onto the whole of CM_k to get the boundary operator in degree k.*

Alternative notations are $\mathrm{CM}_\bullet(M; f, g)$ *and* $\mathrm{CM}_\bullet(M; f)$.

This definition seems rather abstract, however it is in fact only a more analytical reformulation of cellular homology. Let (M, g) be a compact Riemmanian manifold. As explained in the first chapter of [2], using a Morse function $f : M \to \mathbb{R}$, one can construct a handlebody decomposition of M. This is done by looking at the sublevelsets $M^a = f^{-1}((-\infty, a])$. If the interval $[a, b]$ does not contain a critical value of f, then the negative gradient flow of f induces a retraction, which is also a diffeomorphism $M^a \cong M^b$. If, on the other hand, $f(x) \in [a, b]$ is a critical value of f and no other critical point of f maps into $[a, b]$, then, as far as the homotopy type is concerned, M^b is obtained from M^a by attaching a $\mu(x)$-cell. Unfortunately though, the attaching maps for this decomposition are in general not cellular, so not very useful for computing homology. This is because when sweeping through the sub-levelsets, critical points are not necessarily traversed in increasing order of their indices, so a lower dimensional cell might attach to a higher dimensional cell. However, there are ways to fix this. One possible route is to use **self-indexing** Morse function, i.e. Morse functions such that for each $x \in \mathrm{Crit}(f)$ we have $f(x) = \mu(x)$. Then cells only attach to lower dimensional ones, and one can even prove that this handlebody construction is a CW-structure on M. This then makes it possible to recover its homology via cellular homology. In fact, the CW-structure is given by the unstable manifolds $W^u(x)$ for $x \in \mathrm{Crit}(f)$. So the fact that $\mathrm{CM}_\bullet$ is generated by $\mathrm{Crit}_k(f)$ in degree k reflects the fact that the CW decomposition has $|\mathrm{Crit}_k(f)|$ cells of dimension k. It even turns out that $[x, y]$ is the degree of the attaching map of the cell represented by x onto the one represented by y. So we even have that as chain complexes:

$$\mathrm{CM}_\bullet(M; f) \cong C^{\mathrm{cell}}(M)$$

We did all this for self-indexing Morse functions so far. However, for general Morse functions with (f, g) nice the above still holds. If (f, g) is Morse-Smale, cells can only connect to lower dimensional ones, so we have a CW structure. This cell complex structure avoids the problem of disorderly traversal encountered with the handlebody decomposition, since cells get sorted by the index of their corresponding critical point. Even though the handlebody decomposition might not be cellular, the decomposition into unstable manifolds still lends itself well for computing the homology. In some sense, Morse homology is a way to take this geometric intuition, and strip away all the geometry, so that only functional analysis remains. This has for instance the advantage that computations can be performed mechanically without the appeal to intuition. Also, if the machinery is developed without the need for a cell complex, it can potentially be applied in cases where there simply is no cell complex. An example would be Floer homology, where the closed manifold M is replaced by a loop space, and the Morse function becomes a functional. The upshot is that in our case, where everything is finite dimensional and well-behaved, one can equivalently think about $\mathrm{CM}_\bullet$ as representing a cellular chain complex.

Before we can proceed to the non-trivial proof that $\mathrm{CM}_\bullet$ as defined here is in fact a chain complex (meaning that $\partial^2 = 0$), we need to formalize the intuition behind example 1.27, that the boundary of $M^f_{x,z}$is composed of broken trajectories. As it stands, the trajectory space is a manifold without boundary. We need to manually glue boundary points to it via the gluing map $\sharp_\rho$. In other words, if theorem 1.26 tells us that a certain trajectory in $M^f_{x,z}$ splits into two trajectories u and v in $M^f_{x,y}$ and $M^f_{y,z}$ respectively, then gluing will adjoin the boundary point (u, v) to the trajectory space. This is made precise by the following theorem:

Theorem 1.31 *Consider critical points such that $k = \mu(x) - 1 = \mu(y) = \mu(z) + 1$. Given a compact set of simply broken trajectories $K \subset M^f_{x,y} \times M^f_{y,z}$, there is a smooth map:*

$$\begin{aligned} \sharp : K \times (0, 1] &\to M^f_{x,z} \\ (u, v, \rho) &\mapsto u \sharp_\rho v \end{aligned}$$

such that the map $\sharp_\rho$ is an embedding for each choice of ρ. Moreover this family of maps induces a smooth embedding:

$$\hat{\sharp} : \hat{M}^f_{x,y} \times \hat{M}^f_{y,z} \times (0, 1] \to \hat{M}^f_{x,z}$$

such that we obtain weak convergence towards to a simply broken trajectory:

$$\hat{u} \hat{\sharp}_\rho \hat{v} \xrightarrow{C^\infty_{loc}} (\hat{u}, \hat{v})$$

as the gluing parameter ρ tends towards 0. Moreover we have the converse result, namely that any sequence of unparametrized trajectories converging to a simply broken trajectory finally lies within the range of such a gluing map $\hat{\sharp}$.

Using this theorem we can now think of $\hat{M}^f_{x,z}$ as a manifold with boundary, where $\partial \hat{M}^f_{x,z} = \bigsqcup_{y \in \mathrm{Crit}_k(f)} \hat{M}^f_{x,y} \times \hat{M}^f_{y,z}$. Before adding boundary points, the only obstruction to compactness of the trajectory space was the convergence to broken trajectories. However now that we added them to the space, we have effectively compactified it, since every sequence in the extended space clearly has a convergent subsequence in the extended space. We now have all the tools we need to tackle the proof of

Proposition 1.32 $(\mathrm{CM}_\bullet, \partial)_{f,g}$ *is indeed a chain complex, i.e. $\partial^2 = 0$.*

Proof It suffices to prove that ∂^2 vanishes on basis elements, since $(\mathrm{CM}_\bullet)_k$ is free. Consider therefore some critical point in $x \in \mathrm{Crit}_k(f)$. Then

$$\begin{aligned}
\partial^2 x &= \partial \sum_{y \in \mathrm{Crit}_{k-1}(f)} [x,y]y \\
&= \sum_{y \in \mathrm{Crit}_{k-1}(f)} [x,y]\partial y \\
&= \sum_{y \in \mathrm{Crit}_{k-1}(f)} \sum_{z \in \mathrm{Crit}_{k-2}(f)} [x,y][y,z]z \\
&= \sum_{z \in \mathrm{Crit}_{k-2}(f)} \left(\sum_{y \in \mathrm{Crit}_{k-1}(f)} [x,y][y,z] \right) z
\end{aligned}$$

so in fact it suffices to show that $\sum_{y \in \mathrm{Crit}_{k-1}(f)} [x,y][y,z] = 0$ for all $x \in \mathrm{Crit}_k(f)$ and $z \in \mathrm{Crit}_{k-2}(f)$. Notice that by definition, $\mathrm{Crit}_l(f) =$ for $l < 0$, hence the cases $k = 0$ and $k = 1$ are trivial. So consider $k > 2$ from now on.

From the discussion above we know that $\hat{M}^f_{x,z}$ can be thought of as a compact 1-manifold with boundary $\partial \hat{M}^f_{x,z} = \bigsqcup_{y \in \mathrm{Crit}_k(f)} \hat{M}^f_{x,y} \times \hat{M}^f_{y,z}$. From the classification of 1-manifolds, we know that $\hat{M}^f_{x,z}$ is diffeomorphic to a finite union of circles and closed intervals, hence its boundary has an even number of points. But

$$\begin{aligned}
0 &\equiv |\partial \hat{M}^f_{x,z}| && \mod 2 \\
&\equiv \sum_{y \in \mathrm{Crit}_{k-1}(f)} |\hat{M}^f_{x,y}||\hat{M}^f_{y,z}| && \mod 2 \\
&\equiv \sum_{y \in \mathrm{Crit}_{k-1}(f)} [x,y][y,z] && \mod 2
\end{aligned}$$

by definition. So we are done. □

As the Morse chain complex is indeed a chain complex, we can take its homology, which we call:

Definition 1.33 *The* ***k-th Morse Homology group with*** $\mathbb{Z}_2$ ***coefficients*** *of a smooth manifold with respect to a nice pair* (f,g) *is defined as:*

$$\mathrm{HM}_k(M; f, g) := H_k((\mathrm{CM}_\bullet, \partial)_{f,g})$$

Example 1.34 *(Torus, Sphere)*

- *The 2-Torus* $\mathbb{T}^2$ *with the above defined Morse function has as its Morse complex*

  $$0 \to \mathbb{Z}_2 \xrightarrow{0} \mathbb{Z}_2^2 \xrightarrow{0} \mathbb{Z}_2$$

  where all the boundary operators are zero, as there always is an even number of trajectories from a degree k *critical point to a degree* $k-1$ *critical point.*

- *Spheres S^n with $n > 0$ where we consider as Morse function the projection to the last coordinate π_{n+1}, have, as previously explained,only two critical points: the north pole of index n and the south pole of index 0. We get the Morse chain complex:*

$$0 \to \mathbb{Z}_2 \overset{0}{\to} \ldots \overset{0}{\to} \mathbb{Z}_2$$

Where the n-th as well as the 0-th group are the only nonzero ones.

These are two examples where the computational power of Morse homology shines, as all the boundary operators vanish for our examples $\mathbb{T}^2$ and S^n where ($n > 0$). This makes the homology trivial to compute. In general, Morse functions with the added property that the boundary operators all vanish are called **perfect***, and allow for instance to compute the homology of grassmannians (see chapter 8 of [1]).*

1.4 Continuation Maps

Now that we have defined Morse Homology with the help of some parameters, namely the Morse function f and the Riemannian metric g, we would like to get rid of these non-canonical choices again. This should be possible, since no matter which cell decomposition is chosen for M, it computes the same homology. In this section we will construct explicit isomorphisms between the Morse Homology theories associated to different nice pairs, which shows that no matter what initial parameters we choose we get the same result. This will then be used in the next section to construct Morse Homology groups that don't presume a specific choice of (f, g).

Let's first outline which maps we are going to consider:

- The $\Phi^{\alpha\beta}$ will be chain maps between

$$(\mathrm{CM}_\bullet(M; f^\alpha, g^\alpha), \partial) \to (\mathrm{CM}_\bullet(M; f^\beta, g^\beta), \partial).$$

- The $\varphi_k^{\alpha\beta}$ will be the induced maps in degree k homology of the $\Phi^{\alpha\beta}$.
- The $\Psi^{\alpha\beta\gamma}$ will be chain homotopies between $\Phi^{\beta\gamma} \circ \Phi^{\alpha\beta}$ and $\Phi^{\alpha\gamma}$. It turns out that $\Phi^{\alpha\alpha}$ will be the identity on the chain level.

The existence and properties of these maps are the main result of this section. From it follows a very important corollary, which is the first step in proving the functoriality of Morse homology:

Theorem 1.35 *The induced map in homology*

$$\varphi_k^{\alpha\beta} : \mathrm{HM}_k(M; f^\alpha, g^\alpha) \to \mathrm{HM}_k(M; f^\beta, g^\beta)$$

is an isomorphism of groups.

Proof The existence of chain homotopies $\Psi^{\alpha\beta\gamma}$ implies that in homology the maps $H_k(\Phi^{\beta\gamma} \circ \Phi^{\alpha\beta}) = H_k(\Phi^{\beta\gamma}) \circ H_k(\Phi^{\alpha\beta}) = \varphi_k^{\beta\gamma} \circ \varphi_k^{\alpha\beta}$ and $H_k(\Phi^{\alpha\gamma}) = \varphi_k^{\alpha\gamma}$ are identical. So in particular

$$\varphi_k^{\beta\alpha} \circ \varphi_k^{\alpha\beta} = \varphi_k^{\alpha\alpha} = \mathrm{id}$$

in homology. From this we conclude that $\varphi_k^{\alpha\beta}$ has a left inverse. Swapping the roles of α and β we see it also has a right inverse, which is the same as its right inverse. Hence $\varphi_k^{\alpha\beta}$ is an isomorphism of groups. □

The rest of this section will focus on constructing the $\Phi^{\alpha\beta}$ and $\Psi^{\alpha\beta\gamma}$ and show that they have the postulated properties. First, the chain maps between different Morse complexes. Here, the link will be special homotopies between the Morse function and similarly for the metrics.

Definition 1.36

- *A* ***Morse homotopy*** *is a homotopy* $h^{\alpha\beta} : \mathbb{R} \times M \to \mathbb{R}$ *between Morse functions* f^α *and* f^β *such that the following is satisfied:*

  1. $h^{\alpha\beta}$ *is* ***finite****, i.e.*

  $$h^{\alpha\beta}(t, \cdot) = \begin{cases} f^\alpha, & t \leq -1 \\ f^\beta, & t \geq 1 \end{cases}$$

  2. $h^{\alpha\beta}$ *is* ***regular****. This is a technical condition on the critical points of* $h^{\alpha\beta}$*, which is generically satisfied. The exact definition can be found in [3].*

  3. $\lim_{n\to\infty} h^{\alpha\beta}(t_n, x_n) = \infty$*, if* $(t_n, x_n)_{n\in\mathbb{N}} \subset \mathbb{R} \times M$ *and* $d(x_0, x_n) \to \infty$

- *The* ***Homotopy trajectory space*** *is defined analogously to the usual trajectory space. Given a Morse homotopy* $h^{\alpha\beta}$ *between Morse functions* f^α *and* f^β*, a homotopy* g_t *of the metrics* g^α *and* g^β *associated to* f^α *and* f^β *respecively, and critical points* $x \in \mathrm{Crit}(f^\alpha)$ *and* $y \in \mathrm{Crit}(f^\beta)$ *we pose:*

$$M_{x,y}^{h^{\alpha\beta}} := \{\gamma : \dot{\gamma}(t) = \nabla_{g_t} h^{\alpha\beta}(t, \gamma(t)), \lim_{t\to-\infty} \gamma(t) = x, \lim_{t\to+\infty} \gamma(t) = y\}$$

Notice that we do not include the metric in the notation to not further complicate it. Now with these definitions at hand, we can play essentially the same game we played with the parametrized trajectory spaces. As ultimately, we would like to define $\Phi^{\alpha\beta}$ in a similar fashion to the boundary operator $\partial_{(f,g)}$, we need to make sure that there always is a Morse homotopy connecting two Morse functions, and that we can again count a certain kind of flow lines in a reasonable way. The first problem is addressed by the following:

Theorem 1.37 *If M is a closed manifold, and* f^α, f^β *Morse on M, then Morse homotopies between them are generic, and in particular exist.*

Remark 1.38 *This is due to the connectedness of the space of smooth functions and metrics on a manifold, and the fact that Morse homotopies are generic. Hence we can start from any homotopy, and perturb it arbitrary amount to get a Morse homotopy.*

Theorem 1.39 *Given a Morse homotopy $h^{\alpha\beta}$ between two Morse functions f^{α} and f^{β}, then there are generic metrics g^{α}, g^{β} and a generic set of metric homotopies g_t such that $M_{x,y}^{h^{\alpha\beta}}$ is a smooth manifold of dimension $\mu(x) - \mu(y)$.*

A result similar to the compactness result for parametrized trajectory spaces also holds for homotopy trajectory spaces:

Theorem 1.40 *Let $(u_n)_{n\in\mathbb{N}} \subset M^{h^{\alpha\beta}}_{x_\alpha^0, x_\beta^l}$ be a sequence of $h^{\alpha\beta}$-trajectories. Provided that there is no convergent subsequence, there exist critical points:*

$$x_\alpha^0, ...x_\alpha^k = x_\alpha \in \mathrm{Crit}(f^\alpha), x_\beta = x_\beta^0, ..., x_\beta^l \in \mathrm{Crit}(f^\alpha),$$

with $1 \leq k + l \leq \mu(x_\alpha) - \mu(x_\beta)$ and $h^{\alpha\beta}$-trajectories

$$v_\alpha^i \in M^{f^\alpha}_{x_\alpha^i, x_\alpha^{i+1}},\ v_\beta^j \in M^{f^\beta}_{x_\beta^j, x_\beta^{j+1}},\ v_{\alpha\beta} \in M^{h^{\alpha\beta}}_{x_\alpha, x_\beta}$$

together with reparametrization times

$$(\tau^i_{\alpha,n})_{n\in\mathbb{N}}, (\tau^j_{\beta,n})_{n\in\mathbb{N}} \subset \mathbb{R}$$

such that up to the choice of a respectively suitable subsequence weak convergence of the form:

$$u_n \bullet \tau^i_{\alpha,n} \xrightarrow{C^\infty_{loc}} v_\alpha^i,\ u_n \bullet \tau^j_{\beta,n} \xrightarrow{C^\infty_{loc}} v_\beta^j,\ u_n \xrightarrow{C^\infty_{loc}} v_{\alpha\beta}$$

with $0 \leq i \leq k - 1$ and $0 \leq j \leq l - 1$ holds. Moreover we have:

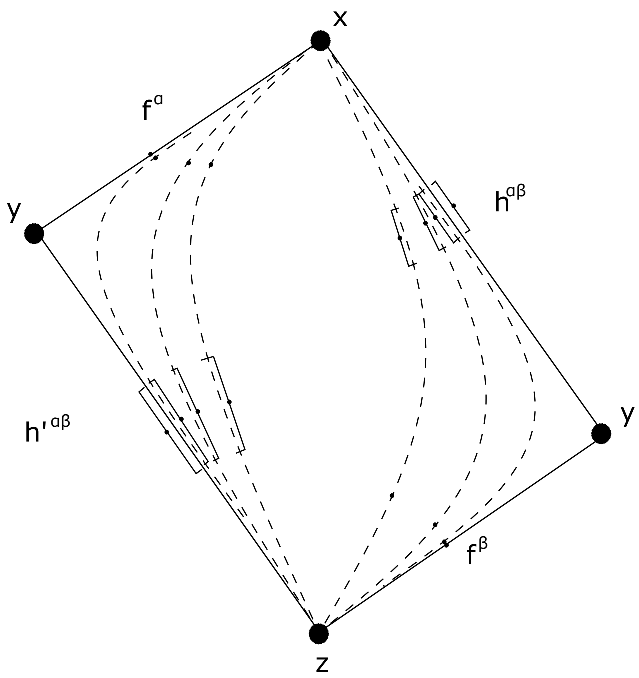
$$\mu(x_\alpha^0) < ... < \mu(x_\beta^k) < ... < \mu(x_\beta^l)$$

Proof The proof is at its core the same as for theorem 1.26, where we showed that $M^f_{x,y}$ is compact up to broken trajectories. However in this case, the equicontinuity needed for Arzela-Ascoli needs the further assumption that the homotopy is finite. What is needed is the compactness of the set on which $\partial_s h^{\alpha\beta} \neq 0$. In the breaking step, we get the following two cases:

- The limit is in $M^{h_{\alpha\beta}}_{x_\alpha^0, x_\beta^l}$ again, then we are done.

- The limit is in some $M_{x,y}^{h_{\alpha\beta}}$ different from $M_{x_\alpha^0,x_\beta^l}^{h_{\alpha\beta}}$. Then we can again choose times τ_n diverging to $-\infty$ such that if we reparametrize u_n as $\tilde{u}_n = u_n \bullet \tau_n$, then it's not necessarily a homotopy trajectory again, but on every compact subset $K \subset \mathbb{R}$ the family $\tilde{u}_n$ will be a flow line of f^α or f^β. This is because $\tau_n \to -\infty$, as this means that eventually, the part where the homotopy happens drops out of K, and hence gets not considered when talking about convergence on compact sets. □

Remark 1.41 *The splitting up of homotopy trajectories can be seen pictorally in the case* $k + l = 1$.



The segment of the trajectory marked by an accolade is the portion where the homotopy kicks in, i.e. were $-1 \le t \le +1$. *Before this part, a* $h^{\alpha\beta}$*-trajectory is just a simple* f^α *trajectory, and after that it's a* f^β *trajectory.*

Again as before we have an immediate

Corollary 1.42 *If* $x \in \mathrm{Crit}_k(f^\alpha)$ *and* $y \in \mathrm{Crit}_k(f^\beta)$ *then* $M_{x,y}^{h^{\alpha\beta}}$ *is a finite set.*

Furthermore there is a similar result if $\mu(x) - \mu(z) = 1$ for $x \in \mathrm{Crit}(f^\alpha)$ and $z \in \mathrm{Crit}(f^\beta)$.

Corollary 1.43 *If* $x \in \mathrm{Crit}_{k+1}(f^\alpha)$ *and* $z \in \mathrm{Crit}_k(f^\beta)$, *then* $M_{x,z}^{h_{\alpha\beta}}$ *is a smooth one-dimensional manifold without boundary, that can be compactified by adding as bound-*

ary points the broken trajectories of theorem 1.40. This is done using a gluing construction similar to the trajectory spaces for Morse functions. The compactified manifold $\bar{M}^{h_{\alpha\beta}}_{x,z}$ *satisfies:*

$$\partial \bar{M}^{h_{\alpha\beta}}_{x,z} = \bigcup_{y \in \mathrm{Crit}_k(f^\alpha)} M^{f^\alpha}_{x,y} \times M^{h^{\alpha\beta}}_{y,z} \\ \cup \bigcup_{y' \in \mathrm{Crit}_{k+1}(f^\beta)} M^{h^{\alpha\beta}}_{x,y'} \times M^{f^\beta}_{y',z}$$

We are now able to define

$$[x,y]_{h^{\alpha\beta}} := |M^{h_{\alpha\beta}}_{x,y}| \mod 2.$$

Next we can define the chain maps that will link the Morse complexes for different nice pairs, if we are given a Morse homotopy between them.

Definition 1.44 *Given two nice pairs* (f^α, g^α) *and* (f^β, g^β) *as well as a Morse homotopy* $h^{\alpha\beta}$ *between them we define the* ***continuation map*** *:*

$$\Phi^{\alpha\beta} : \mathrm{CM}_\bullet(M; f^\alpha, g^\alpha) \to \mathrm{CM}_\bullet(M; f^\beta, g^\beta)$$

by positing for critical points $x \in \mathrm{Crit}_k(f^\alpha)$*:*

$$\Phi^{\alpha\beta}(x) = \sum_{y \in \mathrm{Crit}_k(f^\beta)} [x,y]_{h^{\alpha\beta}} y$$

and extend onto formal sums of critical points by linearity.

In particular if the two pairs agree, one can check that the constant homotopy between them is Morse, from which it readily follows that $\Phi^{\alpha\alpha}$ is the identity on the chain level.

Proposition 1.45 *The map* $\Phi^{\alpha\beta}$ *is a well defined chain map.*

Proof We need to prove $\partial \circ \Phi^{\alpha\beta} = \Phi^{\alpha\beta} \circ \partial$, and it suffices to show it for $x \in \mathrm{Crit}_k(f^\alpha)$. We have because we work over $\mathbb{Z}_2$:

$$\begin{aligned}
(\partial \circ \Phi^{\alpha\beta} - \Phi^{\alpha\beta} \circ \partial)x &= (\partial \circ \Phi^{\alpha\beta} + \Phi^{\alpha\beta} \circ \partial)x \\
&= \partial \sum_{y \in \mathrm{Crit}_k(f^\beta)} [x,y]_{h^{\alpha\beta}} y + \Phi^{\alpha\beta} \sum_{y' \in \mathrm{Crit}_{k-1}(f^\alpha)} [x,y']y' \\
&= \sum_{y \in \mathrm{Crit}_k(f^\beta)} \sum_{z \in \mathrm{Crit}_{k-1}(f^\beta)} [x,y]_{h^{\alpha\beta}} [y,z] z \\
&+ \sum_{y' \in \mathrm{Crit}_{k-1}(f^\alpha)} \sum_{z \in \mathrm{Crit}_{k-1}(f^\beta)} [x,y'][y',z]_{h^{\alpha\beta}} z \\
&= \sum_{z \in \mathrm{Crit}_{k-1}(f^\beta)} \left(\sum_{y \in \mathrm{Crit}_k(f^\beta)} [x,y]_{h^{\alpha\beta}} [y,z] + \sum_{y' \in \mathrm{Crit}_{k-1}(f^\alpha)} [x,y'][y',z]_{h^{\alpha\beta}} \right) z
\end{aligned}$$

Hence it remains to show that the coefficients of z vanish for all $z \in \mathrm{Crit}_{k-1}(f^{\beta})$. By definition:

$$\begin{aligned}
&\sum_{y \in \mathrm{Crit}_k(f^{\beta})} [x,y]_{h^{\alpha\beta}}[y,z] + \sum_{y' \in \mathrm{Crit}_{k-1}(f^{\alpha})} [x,y'][y',z]_{h^{\alpha\beta}} \\
&= \sum_{y \in \mathrm{Crit}_k(f^{\beta})} |M_{x,y}^{h_{\alpha\beta}}||\hat{M}_{y,z}^{f^{\beta}}| + \sum_{y' \in \mathrm{Crit}_{k-1}(f^{\alpha})} |\hat{M}_{x,y'}^{f^{\alpha}}||M_{y',z}^{h_{\alpha\beta}}| \\
&= |\bar{M}_{x,y}^{h^{\alpha\beta}}| \\
&\equiv 0 \mod 2.
\end{aligned}$$

since a compact 1-manifold necessarily has an even number of boundary points. □

Next, we construct explicit chain homotopies between $\Phi^{\beta\gamma} \circ \Phi^{\alpha\beta}$ and $\Phi^{\alpha\gamma}$. To this end we will utilize a final version of the trajectory spaces, the so called λ-parametrized trajectory spaces, which can be thought of as "homotopies between homotopies".

Definition 1.46

- *Assume we have two Morse homotopies $h_0^{\alpha\beta}$ and $h_1^{\alpha\beta}$ together with their associated Riemannian metrics. A **λ-homotopy** is a homotopy of the form :*

$$H^{\alpha\beta} : [0,1] \times \mathbb{R} \times M \to \mathbb{R}$$

  which satisfies the following conditions

  1. $H^{\alpha\beta}(\lambda, t, \cdot) = \begin{cases} f_{\alpha}, & t \leq -1 \\ f_{\beta}, & t \geq 1 \end{cases}$

  2. $H^{\alpha\beta}(i, \cdot, \cdot) = h_i^{\alpha\beta} \quad i \in \{0,1\}$

- *The **λ-homotopy trajectory space** is defined associated to a pair of Morse homotopies as above and critical points $x \in \mathrm{Crit}(f^{\alpha})$ and $y \in \mathrm{Crit}(f^{\beta})$ is defined as:*

$$M_{x,y}^{H^{\alpha\beta}} := \{(\lambda, \gamma) : \dot{\gamma}(t) = \nabla_{g_{\lambda,t}} H^{\alpha\beta}(\lambda, t, \gamma(t)), \lim_{t \to -\infty} \gamma(t) = x, \lim_{t \to +\infty} \gamma(t) = y\}$$

  where $g_{\lambda,t}$ is a homotopy indexed by λ of the homotopies of the Riemmanian metrics associated to $h_i^{\alpha\beta}$.

Theorem 1.47 *Let $h_0^{\alpha\beta}$ and $h_1^{\alpha\beta}$ be Morse homotopies together with their associated Riemmanian metrics. Then there is a generic set of λ-homotopies and a generic set of suitable homotopies of the Riemannian metric, such that $M_{x,y}^{H^{\alpha\beta}}$ is a $(\mu(x) - \mu(y) - 1)$-dimensional manifold.*

Proposition 1.48 *If we are in the situation of the previous theorem and also $\mu(x) - \mu(y) - 1 = 0$, then $M^{H^{\alpha\beta}}_{x,y}$ is compact.*

We will now use the previous theorem to show that no matter which Morse homotopy one chooses between nice pairs, the resulting chain maps are chain homotopic. Let h_0 and h_1 be Morse homotopies between the nice pairs (f^α, g^α) and (f^β, g^β). Then we can find by the previous theorem a λ-homotopy H connecting them, and in particular, for every choice of $x \in \mathrm{Crit}_k(f^\alpha)$ and $z \in \mathrm{Crit}_{k+1}(f^\beta)$ we can find a suitable homotopy of the Riemmanian metric to guarantee that $M^H_{x,z}$ is finite. We use this to define a chain operator:

$$\Psi_k : \mathrm{CM}^\alpha_k \to \mathrm{CM}^\gamma_{k+1}$$

by defining $\Psi(x) = \sum_{z \in \mathrm{Crit}_{k+1}(f^\beta)} |M^H_{x,z}| z$ for $x \in \mathrm{Crit}_k(f^\alpha)$, and extending by linearity. We are then lead to the following

Proposition 1.49 *The chain operator Ψ is in fact a chain homotopy between $\Phi^{\alpha\beta}_0$ and $\Phi^{\alpha\beta}_1$ associated to h_0 and h_1 respectively.*

Proof What needs to be shown is that

$$\Theta := \partial\Psi + \Psi\partial - \Phi^{\alpha\beta}_0 + \Phi^{\alpha\beta}_1 = 0$$

As before this can be reduced to the case where we apply both sides to a basis element $x \in \mathrm{Crit}_k(f^\alpha)$. Now in exactly the same manner as done before, we can massage this expression to be of the form:

$$\Theta x = \sum c_z z$$

where $c_z \in \mathbb{Z}_2$. Finally, these c_z can be shown to be even, by equating them to the number of boundary points of $M^H_{x,z}$. □

Now we are in good shape to define the chain homotopies needed.

Definition 1.50 *Let $h^{\alpha\beta}$, $h^{\beta\gamma}$ and $h^{\alpha\gamma}$ be Morse homotopies between Morse functions f^α, f^β, f^γ and define $h_0 := h^{\alpha\gamma}$ as well as the concatenation $h_1 := h^{\alpha\beta} \star h^{\beta\gamma}$. We can then consider the following λ-homotopy between h_0 and h_1:*

$$H(\lambda, t, p) = (1 - \lambda)h_0(t, p) + \lambda h_1(t, p)$$

Now choose appropriate λ-homotopies of the metric, so that for $x \in \mathrm{Crit}_k(f^\alpha)$, $z \in \mathrm{Crit}_k(f^\gamma)$ and $\mu(x) - \mu(z) - 1 = 0$ we have that $M^H_{x,z}$ is compact. Then consider the map:

$$\Psi^{\alpha\beta\gamma} : \mathrm{CM}^\alpha_k \to \mathrm{CM}^\gamma_{k+1}$$

where the C_n^γ *are the groups in degree n of the Morse chain complex associated to* f^γ*, defined on basis elements* $x \in \mathrm{Crit}_k(f^\alpha)$ *by :*

$$\Psi^{\alpha\beta\gamma}(x) = \sum_{z \in \mathrm{Crit}_{k+1}(f^\gamma)} |M^H_{x,z}| z$$

and extend onto the rest by linearity.

This is again well-defined by the same arguments as the previous times. As an immediate consequence of the previous result we get that:

Proposition 1.51 $\Psi^{\alpha\beta\gamma}$ *is a chain homotopy between* $h_0^{\alpha\gamma}$ *and* $h_1^{\alpha\gamma}$*, where we are in the setting of the previous definition.*

Remark 1.52 *It is interesting to note that the three proofs that*

$$M^f_{x,y}, \quad M^{h^{\alpha\beta}}_{x,y} \quad \text{and} \quad M^H_{x,y}$$

are smooth manifolds can be carried out by the same methods. Using Fredholm theory, these three results can be carried out together. In the same vein the results:

- $\partial^2 = 0$.
- $\Phi^{\alpha\beta}$ *is a chain map.*
- $\Psi^{\alpha\beta\gamma}$ *is a chain homotopy.*

have been proven by exactly the same methodology. This streamlining of proofs is another advantage of modern Morse homology over the classical one.

1.5 Morse Homology

Morse Homology does not really depend on the parameters we used to construct it, in the sense that any choice of a nice pair (f, g) yield some $\mathrm{HM}_k(M; f, g)$ in the same isomorphism class of groups. It is time to employ an algebraic trick, and define the absolute Morse homology groups without reference to any particular nice pair. Morally speaking we want to take all these different representations, multiply them together and consider the diagonal. Formally, this will be done as a limit.

Definition 1.53 *Given a closed manifold M, the category* $\mathbf{NP}(M) = \mathbf{NP}$ *is given by the following data:*

- $\mathrm{obj}(\mathbf{NP}) = \{(f, g) : (f, g) \text{ is a nice pair }\}$
- $\mathrm{hom}_{\mathbf{NP}}((f, g), (f', g')) = \{*_{(f,g),(f',g')}\}$
- *Composition is given by* $*_{(f',g'),(f'',g'')} \circ *_{(f,g),(f',g')} = *_{(f,g),(f'',g'')}$

In more explicit terms, we want the category $\mathbf{NP}(M)$ to have as elements all possible nice pairs, so in particular it is a small category. Furthermore it is non empty, since every closed manifold admits a nice pair. The morphism set between any two objects should contain a single distinguished element $*_{(f,g),(f',g')}$, where the subscript indicates in which morphism set an arrow lives. For $(f,g) = (f',g')$ we identify $\mathrm{id}_{(f,g)} = *_{(f,g),(f,g)}$, so that we get a well defined category. Furthermore, every arrow has a well-defined inverse, hence **NP** is a groupoid category.

For every $k \in \mathbb{N}$ we are now able to define a diagram $T_k : \mathbf{NP} \to \mathbf{Ab}$. We set $T_k(f,g) = \mathrm{HM}_k(M; f,g)$ and $T_k(*_{(f,g),(f',g')}) = \varphi_k^{(f,g),(f',g')}$, where $\varphi_k^{(f,g),(f',g')}$ is induced by some fixed Morse homotopy. Notice that by proposition 1.49 we know that the $\varphi_k^{(f,g),(f',g')}$ are independent of the Morse homotopy chosen between them, and that by remark 1.37, we can always find such a Morse homotopy, as M is compact.

Definition 1.54 *The* ***k-th (independent)$\mathbb{Z}_2$-Morse homology group*** *of a manifold M is defined to be:*

$$\mathrm{HM}_k(M) = \varprojlim T_k = \varprojlim_{(f,g)} \mathrm{HM}_k(M; f,g)$$

As we will always be working with $\mathbb{Z}_2$-coefficients, we will suppress the coefficient group from the notation. Limits need not exist for arbitrary diagrams, but it turns out if the index category is a groupoid category, they do:

Proposition 1.55 *Suppose we are given a diagram $T : \mathbf{J} \to \mathbf{Ab}$, where $\mathbf{J}$ is a small category. Then its limit exists and is given by:*

$$\varprojlim T = G := \{\langle g_\eta : \eta \in \mathbf{J}\rangle : \mathbf{J}(*_{\mu\nu})g_\mu = g_\nu\} \leq \prod_{\alpha \in \mathbf{J}} \mathbf{J}(\alpha)$$

Notice that it is important for the index category to be small, so that we can define the product of groups (in general, a product of groups over a proper class is not again a group, consider for instance $\prod_{x \in V} F(x)$, where V is the Von Neumann Universe and $F(x)$ is the free group with basis x).

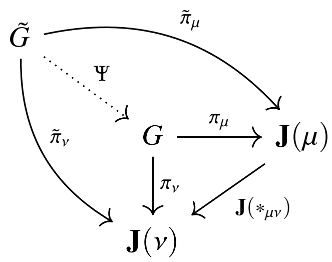
Proof Define $\pi_\mu : G \to \mathbf{J}(\mu)$ to be the projection on the μ-coordinate. Then for any two $\mu, \nu \in \mathbf{J}$ we have:

$$(\mathbf{J}(*_{\mu\nu}) \circ \pi_\mu)(\langle g_\eta : \eta \in \mathbf{J}\rangle) = \mathbf{J}(*_{\mu\nu})(g_\mu) = g_\nu = \pi_\nu(\langle g_\eta : \eta \in \mathbf{J}\rangle)$$

hence

$$\begin{array}{ccc} G & \xrightarrow{\pi_\mu} & \mathbf{J}(\mu) \\ & \searrow^{\pi_\nu} & \downarrow{\scriptstyle \mathbf{J}(*_{\mu\nu})} \\ & & \mathbf{J}(\nu) \end{array}$$

commutes. Hence G together with the π_μ is a cone. Next, we need to make sure that it's the limit. Let $\tilde{G}$ together with $\tilde{\pi}_\mu$ be another solution. Then define $\Psi : \tilde{G} \to G$ by setting $\Psi(\tilde{g}) := \langle \tilde{g}_\eta : \eta \in \mathbf{J} \rangle$ where $\tilde{g}_\eta = \tilde{\pi}_\eta(\tilde{g})$. Then Ψ is well-defined, since $\mathbf{J}(*_{\mu\nu})\tilde{\pi}_\mu(\tilde{g}) = \tilde{\pi}_\nu(\tilde{g})$ by the assumption that $\tilde{G}$ is a solution. Furthermore Ψ is a homomorphism, since all the $\tilde{\pi}$ are, and Ψ is unique, since $(\Psi(\tilde{g}))_\eta \neq \tilde{\pi}_\eta(\tilde{g})$ would imply $\pi_\eta \circ \Psi \neq \tilde{\pi}_\eta$. Therefore Ψ is the unique homomorphism that makes



commute. Hence $\varprojlim T = G$ as desired. □

As a direct consequence of this proposition and the fact that **NP** is a groupoid category (so every arrow is invertible), we get the following result:

Theorem 1.56 *The k-th $\mathbb{Z}_2$-Morse homology group exists and is given by:*

$$\mathrm{HM}_k(M) = \{\langle g_\eta : \eta \in \mathbf{NP} \rangle : \mathbf{NP}(*_{\mu\nu})g_\mu = g_\nu\} \leq \prod_{(f,g)\in\mathbf{NP}} \mathrm{HM}_k(M; f, g)$$

Furthermore the maps $\mathrm{HM}_k(M) \to \mathrm{HM}_k(M; f, g)$ *obtained via the limit are isomorphisms.*

Next, we will define Morse homology for some further classes of smooth manifolds which are not necessarily compact. In each case, we define a version of $\mathbf{NP}(M)$ tailored to the manifold in question. Notice that to define $\mathrm{HM}_k(M)$ for closed manifolds, we needed $\mathbf{NP}(M)$ to be a small groupoid category and a diagram $T_k : \mathbf{NP}(M) \to Ab$ which sent the symbols in $\mathbf{NP}(M)$ to Morse homology groups and continuation maps between them. Looking back at the regularity and compacteness results, we didn't need the fact that M was compact, only that all the pairs in question should be nice and that the homotopies should be Morse. So if we can show the existence of nice pairs as well as the existence of Morse homotopies, we can proceed in exactly the same way as above. After that, we need to show that in the case of ambiguity, all conflicting definitions yield isomorphic Morse homologies.

Euclidean space $\mathbb{R}^n$

In general, the problem of defining Morse Homology for non-compact manifolds is highly non-trivial, as several problems can arise:

(a) Something goes wrong when defining $\mathrm{HM}_k(M; f)$. There might for instance be no nice pairs, or one of the compactness theorems might break down.

(b) Something goes wrong when defining $\mathrm{HM}_k(M)$. For instance, if f^α, f^β are two Morse functions, it might happen that $\mathrm{HM}_k(M; f^\alpha) \ncong \mathrm{HM}_k(M; f^\beta)$. Consider in this case for instance the following Morse functions on $\mathbb{R}$.

$$\begin{aligned} f^\alpha : \mathbb{R} &\to \mathbb{R} \\ t &\mapsto t^2 \end{aligned}$$

and $f^\beta = -f^\alpha$.

So one needs to be a little careful when dealing with $\mathbb{R}^n$. However there are a few conditions that assure everything works out fine at least for (b). We still have $\mathrm{HM}_k(M; f^\alpha) \cong \mathrm{HM}_k(M; f^\beta)$ in the following cases:

(a) $\|f^\alpha - f^\beta\|_{C^0}$ is finite.

(b) f^α and f^β both satisfy the coercivity condition we introduced in the definition of a nice pair.

(c) There is some $N \subset M$ compact such that $M \setminus N \cong \partial N \times \mathbb{R}$ and all the critical points and flow lines of f^α and f^β run entirely in N^{o}.

(d) $M_{x,y}^{f^\alpha}$ and $M_{u,v}^{f^\beta}$ are compact for any choice of x, y, u, v.

These are all ways to mitigate the possibilities of f being pathological at infinity. We will only ever work with manifolds where we can impose either the second or the third condition. In the case of $\mathbb{R}^n$ we choose as Morse functions all the coercive Morse functions. These for instance include all the quadratic forms $q(x) = \|x - p\|^2$, where $p \in \mathbb{R}^n$ is an arbitrary point. We consider as metrics all the metrics that complete $(f, \cdot)$ to a nice pair. For quadratic forms, we have $\mathrm{Crit}(q) = \{p\}$ and $W^u(p) = \{p\} \pitchfork \mathbb{R}^n = W^s(p)$ in the standard Euclidean metric, so in particular a nice pair exists. Since we're only using coercive functions and the second condition above applies, we will just take as a given that continuation maps exist and satisfy the same identities we proved for compact manifolds. Let us call $\mathbf{NP}_{euc}(\mathbb{R}^n)$ be the corresponding index category. It gives rise to a well defined $\mathrm{HM}_k^{euc}(\mathbb{R}^n)$.

Product manifolds $M \times N$

Choose all the product Morse functions $f \oplus g : M \times N \to \mathbb{R}$ together with the product metrics. It is straightforward to check that nice pairs on M and N again yield nice pairs on $M \times N$ and that Morse homotopies can be combined to yield Morse homotopies of the product Morse functions. We denote the corresponding index category by $\mathbf{NP}_{prod}(M \times N)$ and the corresponding Morse homology by $\mathrm{HM}_k^{prod}(M \times N)$.

Vector bundles over compact manifolds $\pi : E \to M$

As with $\mathbb{R}^n$, we are in non-compact territory, so we must tread carefully. However since we consider vector bundles over compact manifolds, we can apply either the third condition with N being the unit ball bundle associated to π, or only consider coercive functions. Examples of both such types of functions can be given as follows. Fix a Riemmanian metric on E, i.e. a smooth collection of maps $g_p : E_p \times E_p \to \mathbb{R}$ where E_p is the fiber at $p \in M$ of the vector bundle. Consider a Morse function $f : M \to \mathbb{R}$. Then

$$\begin{aligned} f_E &: E \to \mathbb{R} \\ (p, v) &\mapsto f(p) + g_p(v, v) \end{aligned}$$

is a Morse function on E that satisfies $f_E|_M = f$ as well as $\mathrm{Crit}(f_E) \cong \mathrm{Crit}(f)$. This can be seen from the local product structure of the vector bundle together with $\mathrm{Crit}(v \mapsto g_p(v, v)) = \{0_{E_p}\}$. Coercivity is trivially satisfies. If we choose a a Riemmanian metric on TE, such that it respects the vertical-horizontal splitting of tangent bundle and the horizontal component of the metric together with f form a nice pair, then the Morse-Smale condition is also satisfied. In this case, $g_p(v, v)$ decreases along flow lines, so that if $(p, v) \in E \setminus M$, then a flow line passing through (p, v) has to diverge if times goes to negative infinity. This means that $E \setminus M$ is not part of any unstable manifold, so that for $E \setminus M$ transverse intersection of stable and unstable manifolds is guaranteed. Concerning a neighborhood of M, we also have transversality because of the local product structure and the well-behaved metric.

We now collect all these nices pairs together with all the other coercive Morse functions on E to get our new category $\mathbf{NP}_{vec}(E)$. Considering Morse homotopies, for the special class discussed above, we can combine the Morse homotopy used for Euclidean spaces with a Morse homotopy in M using the local product structure to get Morse homotopies in E. This then allows us to define Morse Homology for vector bundles over compact manifolds.

To show that the definitions don't conflict we will use the following property of limits:

Proposition 1.57 *Let $\Lambda' \subset \Lambda$ be directed sets where the inclusion is order-preserving and cofinal. Suppose $T : \mathbf{J}(\Lambda) \to C$ is a diagram, where $\mathbf{J}(\Lambda)$ denotes the category associated to the directed set Λ and C is any category and denote by T' the restriction of T to Λ'. Then*

$$\varprojlim_{\Lambda} T \cong \varprojlim_{\Lambda'} T'$$

The same results holds true, if we consider any injection $\Lambda' \hookrightarrow \Lambda$, and can be translated to our situation, where we work with categories $\mathbf{NP}$ which are implicitly of the form $\mathbf{J}(\Lambda)$ for some Λ. So if for any two collisions, we can find a cofinal order preserving injection from the one $\mathbf{NP}$ into the other, they yield identical Morse homology groups. However, since all the orders we are working with are trivial, we only need to make sure that our map is an injection, since order-preservation and cofinality are satisfied trivially.

Now the different possibilities of conflict are:

- For $n, m \in \mathbb{N}$, we can see $\mathbb{R}^{n+m}$ as Euclidean space, or as a product in a number of ways.
- For compact M, N, we can compute $\mathrm{HM}_k(M \times N)$ via $\mathbf{NP}(M \times N)$ or via $\mathbf{NP}_{prod}(M \times N)$
- For compact M, N, P we can compute $\mathrm{HM}_k(M \times N \times P)$ in different ways via products.

Thankfully, in each case, the appropriate inclusion is easy to find, so that we have proven our definitions not to be contradictory. In the following we will therefore just write $\mathrm{HM}_k(M)$ for all of the constructions above without explicitly mentioning with respect to which version of **NP** the limit is taken.

Chapter 2

Eilenberg-Steenrod axioms

There are a plethora of approaches to distinguish topological spaces with algebraic invariants. A very important family of such invariants are homology theories, which are countable families of functors H_* from categories of spaces to abelian groups. Eilenberg and Steenrod realized that there are a handful of key properties that a lot of them have in common. These few main ingredients are called the Eilenberg-Steenrod axioms, and countable families of functors that satisfies them are called Eilenberg-Steenrod homology theories, or just homology theories for short. From these axioms useful results can be derived, such as the Mayer-Vietoris sequence for instance. In this chapter we will show that Morse homology with $\mathbb{Z}_2$ coefficients is in fact a $\mathbb{Z}_2$-homology theory in the sense of Eilenberg and Steenrod. First however we need to show that the $\mathrm{HM}_*(M)$ as defined in the last chapter can be extended to become functors. We can then go on to defining the relative Homology groups $\mathrm{HM}_*(M,A)$ for admissible pairs (M,A) and verify the Eilenberg-Steenrod axioms. We will then present a uniqueness result which states that all $\mathbb{Z}_2$-ES-theories are isomorphic for non pathological spaces. This will allow us to conclude that:

$$\mathrm{HM}_*(M) \cong \mathrm{H}_*^{sing}(M)$$

for closed manifolds, since it is a well known fact from algebraic topology that singular homology is an ES-theory.

2.1 Functoriality

The task of extending the HM_* to become functors :

$$\mathrm{HM}_* : \mathbf{cMan} \to \mathbf{Ab}$$

where **cMan** is the category of closed smooth manifolds together with smooth maps, can be subdivided into 4 steps, where $\mathrm{HM}_*(f)$ will be defined for different kinds of smooth maps $f : M \to N$:

- Diffeomorphisms $\varphi : M \xrightarrow{\cong} N$
- Closed embeddings $\iota : M \hookrightarrow N$
- Projections : $\pi : M \times N \to M$
- General smooth maps : $\chi : M \to N$

Always keep in mind that in finite dimensional Morse homology, the critical points of a Morse function represent the cells of an implicit cell complex. Correspondingly, maps between manifolds should ideally send cells to other cells in a canonical way. As this is however not always the case, the constructions for diffeomorphisms and closed embeddings consider "matching" Morse functions on both manifolds, which make it easiest to assign cells to cells in a non-ambiguous way. In the later steps, i.e. for projections and smooth maps in general, we will complete the definition by making sure that the induced homomorphims stay functorial.

Diffeomorphisms

Let's first assume we are provided with a diffeomorphism $\varphi : M \xrightarrow{\cong} N$ between two smooth closed manifolds M and N. Choose a Morse function $f : M \to \mathbb{R}$ on M. The first thing to notice is that:

Proposition 2.1 *The pushforward $\varphi_* f := f \circ \varphi^{-1} : N \to \mathbb{R}$ is a Morse function on N and for all $k \in \mathbb{N}$ we have $\varphi(\mathrm{Crit}_k(f)) = \mathrm{Crit}_k(\varphi_* f)$.*

Proof Consider a point $p \in M$. Then by the chain rule:

$$d(\varphi_* f)_{\varphi(p)} = df_p \circ (d\varphi)^{-1}_{\varphi(p)}$$

Now since φ is a diffeomorphism, $d\varphi$ is an isomorphism of vector spaces, so that

$$df_p = 0 \Leftrightarrow d(\varphi_* f)_{\varphi(p)} = 0$$

or in other words $\varphi(\mathrm{Crit}(f)) = \mathrm{Crit}(\varphi_* f)$. Next consider a parametrization $\psi : \Omega \subset \mathbb{R}^m \to U_p \subset M$ of a neighborhood U_p of a critical point $p \in \mathrm{Crit}_k(f)$. We can then consider with respect to this parametrization:

$$M_p(f) = \left(\frac{\partial^2 (f \circ \psi)}{\partial x_i \partial x_j}(p)\right)_{i,j} = \left(\frac{\partial^2 ((f \circ \varphi^{-1}) \circ (\varphi \circ \psi))}{\partial x_i \partial x_j}(p)\right)_{i,j} = M_{\varphi(p)}(\varphi_* f)$$

where $M_{\varphi(p)}(\varphi_* f)$ is defined in terms of the parametrization $\varphi \circ \psi$. Hence p is non-degenerate iff $\varphi(p)$ is non-degenerate, so $\varphi_* f$ is Morse. Furthermore $\mu(p) = \mu(\varphi(p))$ since both hessian matrices have exactly the same eigenvalues. □

Next we notice that for any Riemannian metric g on M:

$$\varphi : (M, g) \to (N, \varphi_* g)$$

is an isometry. It turns out, that if (f, g) is nice, we automatically have that $(\varphi_* f, \varphi_* g)$ is nice as well. This is because the Morse-Smale condition (which is a transversality condition) is preserved under diffeomorphims, as their derivatives induce isomorphisms between tangent spaces. Hence we can define a bijection between chain complexes associated to nice pairs:

$$\varphi_\bullet : \mathrm{CM}_\bullet(M; f, g) \to \mathrm{CM}_\bullet(N; \varphi_* f, \varphi_* g)$$

by sending a critical point $x \in \mathrm{Crit}(f)$ to $\varphi_* f(x) \in \mathrm{Crit}(\varphi_* f)$ and extending by linearity. We would like this to be a chain map so that it induces a well-defined homomorphism in homology. Let's check that it is in fact a chain map. Knowing that φ can without loss of generality be chosen to be isometrical we can relate the trajectories on both manifolds by:

Proposition 2.2 *There is a one to one correspondence under isometrical diffeomorphisms φ between trajectories on M and on N. We have that:*

$$\gamma \in M^f_{x,y} \Leftrightarrow \varphi_*\gamma := \varphi \circ \gamma \in N^{\varphi_* f}_{\varphi(x),\varphi(y)}$$

Proof Consider a trajectory $\gamma \in M^f_{x,y}$, then:

$$\begin{aligned} \frac{d}{dt}(\varphi_*\gamma) =& d\varphi_{\gamma(t)}\dot{\gamma} \\ =& - d\varphi_{\gamma(t)}\nabla_g f(\gamma(t)) \\ =& - \nabla_{\varphi_* g}(\varphi_* f)(\varphi_*\gamma(t)) \end{aligned}$$

Here we have used in the last equality that for any $v \in T_{\varphi(\gamma(t))}N$:

$$\begin{aligned} d(\varphi_* f)_{\varphi_*\gamma(t)} v =& d(f \circ \varphi^{-1})_{\varphi(\gamma(t))} v \\ =& df_{\gamma(t)} \circ (d\varphi)^{-1}_{\varphi(\gamma(t))} v \\ =& g_{\gamma(t)}(\nabla f(\gamma(t)), (d\varphi)^{-1}_{\varphi(\gamma(t))} v) \\ =& \varphi_* g_{\varphi(\gamma(t))}(d\varphi_{\gamma(t)}\nabla f(\gamma(t)), v) \end{aligned}$$

so that $d\varphi_{\gamma(t)}\nabla_g f(\gamma(t))$ satisfies the condition to be the gradiant of $\varphi_* f$ with respect to the metric $\varphi_* g$. This shows that φ maps trajectories to trajectories. Now since:

$$\lim_{t\to\pm\infty} \varphi_*\gamma(t) = \varphi(\lim_{t\to\pm\infty} \gamma(t))$$

we also know that $\varphi_*\gamma \in N^{\varphi_* f}_{\varphi(x),\varphi(y)}$. Furthermore, since φ^{-1} is a diffeomorphism and $(\varphi^{-1})_*\varphi_* g = (\varphi^{-1} \circ \varphi)_* g = g$, the pushforward trough φ maps trajectories on M bijectively to trajectories on N. □

Corollary 2.3 *Given an isometrical diffeomorphism $\varphi : M \xrightarrow{\cong} N$ and a Morse function $f : M \to \mathbb{R}$, we have for critical points $x, y \in \mathrm{Crit}(f)$ that*

$$[x, y]_M = [\varphi(x), \varphi(y)]_N$$

where on N, we count the trajectories of $\varphi_* f$.

In terms of cell complexes, this means that φ together with this choice of Morse functions sends cells to other cells while respecting the ways they are attached to each other.

Proposition 2.4 *The map $\varphi_\bullet$ is a chain map, i.e. $\partial\varphi_\bullet = \varphi_\bullet\partial$.*

Proof Letting $x \in \mathrm{Crit}_k(f)$ suffices. Then:

$$\begin{aligned}\partial\varphi_\bullet x &= \sum_{z\in \mathrm{Crit}_{k-1}(\varphi_* f)} [\varphi_\bullet(x), z]z \\ &= \sum_{\varphi(y)\in \mathrm{Crit}_{k-1}(\varphi_* f)} [\varphi(x), \varphi(y)]\varphi(y) \\ &= \sum_{y\in \mathrm{Crit}_{k-1}(f)} [x, y]\varphi(y) \\ &= \varphi_\bullet\left(\sum_{y\in \mathrm{Crit}_{k-1}(f)} [x, y]y\right) = \varphi_\bullet\partial x\end{aligned}$$

□

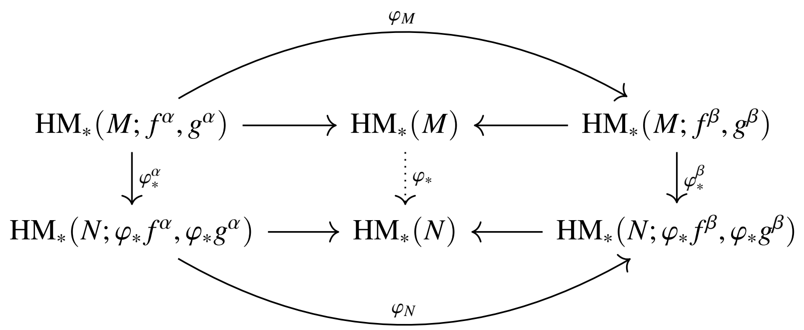
The chain map $\varphi_\bullet$ is a bijection on the chain level, hence

Corollary 2.5 *The homomorphism:*

$$\varphi_* : \mathrm{HM}_*(M; f, g) \to \mathrm{HM}_*(N; \varphi_* f, \varphi_* g)$$

induced by $\varphi_\bullet$ is an isomorphism.

Now that we have an isomorphism between Morse homology groups on both manifolds in dependence of f, we want to make sure that these induce an isomorphism between the limits $\varphi_* : \mathrm{HM}_*(M) \to \mathrm{HM}_*(N)$. In other words, given nice pairs (f^α, g^α) and (f^β, g^β), we would like to determine a unique morphism to put on the dotted arrow:



where φ_M and φ_N are the continuation maps with respect to the homotopies $h^{\alpha\beta}$ and $\varphi_* h^{\alpha\beta}$ (which is a Morse homotopy since φ is a diffeomorphism), and the maps pointing inward are given by the limit. For this to be possible, the diagram without the dot has to commute, which is the case exactly if:

$$\varphi_*^\beta \circ \varphi_M = \varphi_N \circ \varphi_*^\alpha$$

In the same way as before, there is a one to one correspondance between $h^{\alpha\beta}$-trajectories and $\varphi_* h^{\alpha\beta}$-trajectories, and the identity above can be derived in the same way that we derived the fact that $\varphi_\bullet$ is a chain map. So we can define a unique homomorphism φ_* to make the above diagram commute. This construction, i.e. defining a family of homomorphisms for each choice of a nice pair on M and an induced nice pair on N is very useful. The induced pair can be chosen to fit ones needs, and as long as compatibility with the continuation maps is guaranteed, the homomorphisms will descend to the limit. Furthermore, this assignment makes HM_* into a functor

$$\mathrm{HM}_* : \{\text{smooth closed manifolds, diffeomorphisms}\} \to \mathbf{Ab}$$

since

Proposition 2.6 *Given two diffeomorphisms between smooth manifolds $\varphi : M \to N$ and $\psi : N \to P$, we have:*

$$(\psi \circ \varphi)_* = \psi_* \circ \varphi_*$$

Proof On the chain level we have for all Morse functions and any critical point x:

$$(\psi_\bullet \circ \varphi_\bullet)(x) = \psi(\varphi(x)) = (\psi \circ \varphi)_\bullet(x)$$

and this identity also respects Morse homotopies, hence the identity holds also true for the limit. □

Closed embeddings

Let's now suppose we are given a closed embedding $\iota : M \hookrightarrow N$. Fix a Morse function f on M. As above, we get an induced Morse function $\iota_*^0 f$ on $\iota(M)$ via pushforward. Now we'll make use of the following extension result:

Proposition 2.7 *Given a closed submanifold $N \subset M$ of a closed Riemmanian manifold (M, g), as well as a Morse function $f_N \in C^\infty(N, \mathbb{R})$, there is a Morse function $f \in C^\infty(M, \mathbb{R})$ extending f_N:*

$$f|_N = f_N$$

Let g' be any metric on M, such that (f, g') is nice, where f is the extension of f_N constructed above. Then no flow line of f with respect to g' leaves N.

The proof of this proposition is relegated to the appendix. Hence since $\iota(M) \subset N$ is closed we can extend $\iota_*^0 f$ to a Morse function

$$\iota_* f : N \to \mathbb{R}.$$

As in the case with diffeomorphisms, we want the metrics and Morse functions to satisfy some further niceness properties. We choose a metric g_N on N, such that $(\iota_* f, g_N)$ as well as $(f, \iota^* g_M)$ is nice. This we can do, since metrics with $(\iota_* f, g_N)$ nice are generic in $\mathfrak{R}(N)$ as are the ones with $(f, \iota^* g_M)$, since $\iota^* : \mathfrak{R}(N) \to \mathfrak{R}(M)$ is a continuous surjection, so the preimage of dense and open sets is again dense and open. Finally we define $g_M = \iota^* g_N$. These choices having been made, we can now take a look at what they imply. First, $\iota : M \to \iota(M)$ is an isometrical diffeomorphism with our choice of metrics, hence by the previous section there is an isomorphism of chain complexes:

$$\iota^0_\bullet : C_\bullet(M; f, g_M) \to C_\bullet(\iota(M); \iota^0_* f, g_N|_{\iota(M)})$$

Second, since $\mathrm{Crit}(\iota^0_* f) \subset \mathrm{Crit}(\iota_* f)$, we can think of $C_\bullet(\iota(M); \iota^0_* f, g_N|_{\iota(M)})$ as a subset of $C_\bullet(N; \iota_* f, g_N)$.

Third, since flow lines starting at critical points $x \in \mathrm{Crit}(\iota^0_* f) \subset \mathrm{Crit}(\iota_* f)$ stay in $\iota(M)$, we have for $y \in \mathrm{Crit}(\iota_* f)$ that

$$[x, y]_{\iota(M)} = [x, y]_N$$

This means that the boundary operators of $\mathrm{CM}_\bullet(\iota(M))$ and $\mathrm{CM}_\bullet(M)$ agree on critical points in $\iota(M)$. Hence $\mathrm{CM}_\bullet(\iota(M); \iota^0_* f, g_N|_{\iota(M)})$ can even be seen as a subcomplex of $\mathrm{CM}_\bullet(N; \iota_* f, g_N)$ with the natural inclusion map. Combining all of these observations, we are lead to an injective chain map:

$$\iota_\bullet : C_\bullet(M; f, g_M) \to C_\bullet(N; \iota_* f, g_N)$$

which immediately induces a map in homology:

$$\iota_* : \mathrm{HM}_*(M; f, g_M) \to \mathrm{HM}_*(N; \iota_* f, g_N).$$

We would now like to conclude the construction by inducing a map on the absolute groups as such:

$$\begin{array}{ccc} \mathrm{HM}_*(M; f, g_M) & \xrightarrow{\cong} & \mathrm{HM}_*(M) \\ \downarrow \iota^f_k & & \vdots\, \iota_k \\ \mathrm{HM}_*(N; \iota_* f, g_N) & \xrightarrow{\cong} & \mathrm{HM}_*(N) \end{array}$$

As before, to get a map on the absolute homology groups, we need that the map induced on them by ι_* doesn't depend on the choices we made, and is invariant under Morse homotopies. Fortunately though this is the case, we can choose Morse homotopies in such a way that the homotopy trajectories starting in $\iota(M)$ also stay there for any homotopy of the metric. Since the continuation maps don't depend on the homotopy chosen, this shows that the maps we constructed passes to the limit without issue. Functoriality can be checked in the same manner as before. Hence we now have functors:

$$\mathrm{HM}_* : \{\text{closed manifolds, closed embeddings}\} \to Ab$$

Notice that we can extend this construction for closed embeddings to embeddings of the form

$$\begin{aligned} \iota : M &\hookrightarrow \mathbb{R}^k \times M \\ p &\mapsto (c, p) \end{aligned}$$

where $c \in \mathbb{R}^k$ for closed M and $k \in \mathbb{N}$. In this case, we cannot apply the extension result, however we can easily construct an explicit extension of $f \in C^\infty(M, \mathbb{R})$ by setting $\iota^* f = q_c \oplus f$, where $q_c(x) = \|x - c\|^2$ is a positive definite quadratic form. If we then keep in mind that the homology groups are defined by different index categories, the rest of the discussion goes through without issue.

The homotopy lemma

Before proceeding to projections, we need to look at an auxiliary result, which will also later be useful when proving the homotopy axiom for Morse homology.

Lemma 2.8 (Homotopy) *Consider a smooth family* $\{\varphi_t\}^{t\in[0,1]}$ *of closed embeddings between closed manifolds*

$$\varphi^t : M \hookrightarrow N,$$

for which we have already defined the induced maps $\varphi^t_* : \mathrm{HM}_*(M) \to \mathrm{HM}_*(N)$. *Then the* φ^t_* *are independent of the parameter* t.

Proof For any $t \in [0, 1]$, consider the following commuting diagram:

$$\begin{array}{ccc} M & \xrightarrow{\iota^t_M} & \mathbb{R} \times M \\ \downarrow \varphi^t & & \downarrow \psi \\ N & \xrightarrow{\iota^t_N} & \mathbb{R} \times N \end{array}$$

Where

$$\begin{aligned} \iota^t_M &: p \mapsto (t, p) \\ \iota^t_M &: q \mapsto (t, q) \\ \psi &: (t, p) \mapsto (t, \varphi^t(p)) \end{aligned}$$

Since we already have the functorial behaviour for closed embeddings we get the induced diagram in homology:

$$\begin{array}{ccc} \mathrm{HM}_*(M) & \xrightarrow{(\iota^t_M)_*} & \mathrm{HM}_*(\mathbb{R} \times M) \\ \downarrow \varphi^t_* & & \downarrow \psi_* \\ \mathrm{HM}_*(N) & \xrightarrow{(\iota^t_N)_*} & \mathrm{HM}_*(\mathbb{R} \times N) \end{array}$$

By construction, $(\iota^t_M)_*$ and $(\iota^t_N)_*$ are isomorphims. Furthermore they are independent of t, since for $t', t \in [0, 1]$ we have in homology that the following commutes:

$$\begin{array}{ccccccc} \mathrm{HM}_*(M) & \xrightarrow{\cong} & \mathrm{HM}_*(M; f) & \xrightarrow{(\iota^t_M)_*} & \mathrm{HM}_*(\mathbb{R} \times M; q_t \oplus f) & \xrightarrow{\cong} & \mathrm{HM}_*(\mathbb{R} \times M) \\ & & & \searrow^{(\iota^{t'}_M)_*} & \downarrow \tau & & \downarrow id \\ & & & & \mathrm{HM}_*(\mathbb{R} \times M; q_{t'} \oplus f) & \xrightarrow{\cong} & \mathrm{HM}_*(\mathbb{R} \times M) \end{array}$$

where τ is a continuation map and the unnamed arrows arise from the limit definiton of HM_*. From this it is clear that the inclusion maps don't depend on the parameter t in homology. This proves the homotopy lemma. □

This useful lemma leans on the fact that for inclusions ι_M^t and $\iota_M^{t'}$, there is an obvious choice of Morse homotopy between the induced Morse functions. In a sense, the base point of the inclusion gets quotiented out when passing to the limit, hence is irrelevant when dealing with the absolute groups.

Projections

Let's first look at projections of the form:

$$\pi^k : M \times \mathbb{R}^k \to M$$
$$\pi^{k+l,l} : M \times \mathbb{R}^{k+l} \to M \times \mathbb{R}^l$$

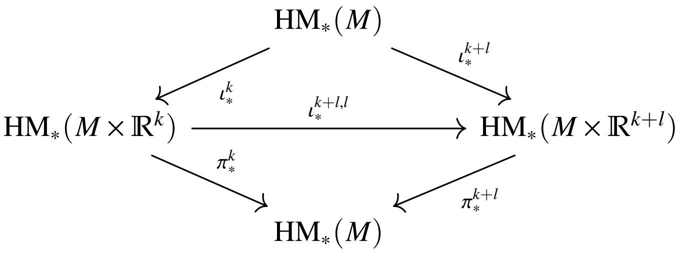
where we just forget about the last k coordinates. These are homotopy inverses of the canonical inclusions

$$\iota^k : M \to M \times \mathbb{R}^k$$
$$\iota^{k+l,l} : M \times \mathbb{R}^l \to M \times \mathbb{R}^{k+l}$$

respectively, where we add trailing zeros. Since $\iota^k : M \to M \times \mathbb{R}^k$ induces an isomorphism on the Morse chain complexes, we also directly get that ι^k induces an isomorphism in homology. Hence

$$\pi_*^k := (\iota_*^k)^{-1} : \mathrm{HM}_*(M \times \mathbb{R}^k) \to \mathrm{HM}_*(M)$$

is also an isomorphism and we can define $p_*^{k+l,l}$ similarly. These definitions are functorially well behaved, as the following commutes :



With this definition, we can proceed to the general case

Definition 2.9 *Given a projection $p : M \times N \to M$ and a closed embedding*

$$\varphi : N \hookrightarrow \mathbb{R}^k$$

for large enough k, we can factor

$$p = \pi^k \circ (id_M, \varphi)$$

Motivated by this decomposition, we set

$$p_* := \pi_*^k \circ (id_M, \varphi)_* \circ : \mathrm{HM}_*(M \times N) \to \mathrm{HM}_*(M)$$

Notice that to be precise, we didn't define the induced map φ_* by the construction above, since $\mathbb{R}^k$ is not compact. However, a similar extension result as the one used for closed embeddings into closed manifolds exists in this special case. This is because $\mathbb{R}^k$ can be seen as a **cone** over $B_1(0) \subset \mathbb{R}^k$, which just means that $\mathbb{R}^k \setminus B_1(0) \cong S^k \times \mathbb{R}$. Informally, this means that $\mathbb{R}^k$ is a well-behaved non-compact manifold, and most of the things we did in the compact case can be modified to also apply here. In conclusion, φ_* can be defined as before and everything works out just fine. However what we need to check now is that this definition doesn't depend on the choice of closed embedding φ.

Proposition 2.10 *The definition of p_* is well-posed.*

Proof Let $\psi : M \hookrightarrow \mathbb{R}^l$ be another closed embedding. In the case that $k \neq l$, we compose φ and ψ with $\iota^{k+l,l}$ and $\iota^{k+l,k}$ respectively to get embeddings into $\mathbb{R}^{k+l}$. By the diagram above, we have $\pi^k_* = \pi^{k+l}_* \circ \iota^{k+l,k}_*$, hence this doesn't change the morphism p_*, so without loss of generality we can assume $k = l$. Now by the homotopy lemma, if we can find a homotopy between φ and ψ, we are done, since then a smooth homotopy between (φ, id_N) and (ψ, id_N) is evident, and hence they induce the same homomorphism in homology. As a further simplification due to the already proven functoriality, we can assume that $k = 2n$, and that the embedding are of the form $\varphi = (\bar{\varphi}, 0)$ and $\psi = (\bar{\psi}, 0)$, where $\bar{\varphi}, \bar{\psi} : M \hookrightarrow \mathbb{R}^n$ are closed embeddings. This allows us to homotope as folllows :

$$(\bar{\varphi}, 0) \overset{H^1_t}{\cong} (0, \bar{\varphi}) \overset{H^2_t}{\cong} (\bar{\psi}, 0)$$

We choose some smooth $\tau : [0,1] \to [0,1]$ with $0 \le \tau(t) \le 1$ for all t, $\tau(t) = 0$ in a neighborhood of 0 and $\tau(t) = 1$ in a neighboorhood of 1. The homotopies H^1_t, H^2_t are then defined as follows:

$$\begin{aligned} H^1_t : M &\to \mathbb{R}^{2n} \\ p &\mapsto ((1-\tau(t))\bar{\varphi}(p), \tau(t)\bar{\varphi}(p)) \\ H^2_t : M &\to \mathbb{R}^{2n} \\ p &\mapsto (\tau(t)\bar{\psi}(p), (1-\tau(t))\bar{\varphi}(p)) \end{aligned}$$

It can be checked that for each t, these are closed embeddings since in the first n coordinates as well as the last n coordinates they are, hence the homotopy lemma applies and we are done. □

Notice that we needed to go through some trouble to homotope between the two embeddings, since directly homotoping between them by linear interpolation would likely have resulted in loosing the embedding property at some intermediate time.

Smooth maps

Now we are ready to define the behaviour of Morse homology on general smooth functions.

Definition 2.11 *Given a smooth map $\chi : M \to N$, we have the factorisation:*

$$\chi = p \circ (\chi, id_M)$$

where $p : N \times M \to N$ is the natural projection. We then set :

$$\mathrm{HM}_*(\chi) := p_* \circ (\chi, id_M)_*$$

This is well defined, since (χ, id_M) is a closed embedding, the graph embedding of χ. Next, we would like to check that for embeddings and projections, this definition coincides with the ones already given. This is the subject of the following two results:

Proposition 2.12 *Given a closed embedding $\iota : M \hookrightarrow N$, we have:*

$$\iota_* = \mathrm{HM}_*(\iota)$$

Proof Let $\varphi : M \to \mathbb{R}^n$ be a closed embedding. Consider the following non-commutative diagram:

$$\begin{array}{ccccc} M & \xrightarrow{\iota} & N & & \\ \downarrow{\scriptstyle (\iota, id)} & & \downarrow{\scriptstyle \iota^k} & \searrow^{id} & \\ N \times M & \xrightarrow{(id,\varphi)} & N \times \mathbb{R}^k & \xrightarrow{\pi^k} & N \end{array}$$

The right triangle commutes, but the square on the left doesn't. However, the diagram commutes up to homotopy, since

$$(\iota, 0) \overset{H_t}{\cong} (\iota, \varphi)$$

via

$$\begin{aligned} H_t : M &\to N \times \mathbb{R}^k \\ p &\mapsto (\iota(p), t\varphi(p)) \end{aligned}$$

where again H_t is a closed embedding for all $t \in [0, 1]$. From the homotopy lemma and by functoriality of embeddings, we then get that

$$(id_N, 0)_* \circ \iota_* = (id_N \times \varphi)_* \circ (\iota, id_M)_*$$

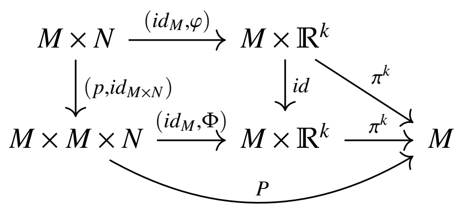
By definition we have $\pi^k_* = (\iota^k_*)^{-1} = (id_N, 0)^{-1}_*$. We follow that:

$$\iota_* = \pi^k_* \circ (id_N \times \varphi)_* \circ (\iota, id_M)_* = \mathrm{HM}_*(\iota) \qquad \square$$

Proposition 2.13 *Given a projection $p : M \times N \hookrightarrow M$, we have:*

$$p_* = \mathrm{HM}_*(p)$$

Proof Let $\varphi : N \to \mathbb{R}^k$ and $\Phi : M \times N \to \mathbb{R}^k$ be a closed embeddings. Consider:



where $P : M \times (M \times N) \hookrightarrow M$ is the obvious projection. Again, the right and lower triangle commute and the sqare doesn't necessarily. However the square can be made to commute up to homotopy by using a construction similar to the one used when showing that the homomorphism induced by a projection is independent of the choice of an embedding, since we're dealing with two closed embeddings with the same domain and codomain. So by the homotopy lemma and functoriality for embeddings we have that the diagram of induced homomorphisms commutes. The upper path

$$p_* = \pi^k_* \circ (id_M, \varphi)$$

hence equals the lower path:

$$\mathrm{HM}_*(p) = P_* \circ (p, id_{M\times N})_* = \pi^k_* \circ (id_M, \Phi) \circ (p, id_{M\times N})_* \qquad \square$$

Theorem 2.14 *Morse homology are well defined functors:*

$$\mathrm{HM}_* : \mathbf{cMan} \to \mathbf{Ab}$$

What is left to check is functoriality, which follows from the fact that projections and embeddings are already functorial, and can be found in $[3, p.150]$.

2.2 Relative Morse homology

The relative homology of a pair $(M, A) \in \mathbf{Top}^2$ considers cycles and boundaries relative to A. Informally, this means that it doesn't matter what happens inside A. Concretely, this intuition can be seen in well behaved spaces. If $(M, A) \in \mathbf{CW}^2$ and A is a subcomplex of M, we have the following:

$$\mathrm{H}^{sing}_*(M, A) \cong \tilde{H}^{sing}_*(M/A)$$

Justification of this fact can be found in $[4, l.19]$. In Morse homology, if one can find Morse functions $f : M \to \mathbb{R}$ and $f_A : A \to \mathbb{R}$ such that $\mathrm{CM}_\bullet(A; f_A)$ can be identified with a subcomplex of $\mathrm{CM}_\bullet(M; f)$, a sensible possibility to remove the dependence on A in $\mathrm{CM}_\bullet(M; f)$ is by looking at the quotient complex:

$$\mathrm{CM}_\bullet(M, A; f, f_A) := \mathrm{CM}_\bullet(M; f)/\mathrm{CM}_\bullet(A; f_A).$$

This would indeed ensure that the critical points inside of A got quotiented out and could not have an impact on the homology of the relative complex. However, to be able to identify $\mathrm{CM}_\bullet(A; f_A)$ with a subcomplex of $\mathrm{CM}_\bullet(M; f)$ a few restrictions on the choice of pairs and Morse functions have to be made.

Definition 2.15 *A pair (M, A) of smooth manifolds without boundary, where A is a submanifold of M, is called* ***admissible*** *if one of the two following conditions hold:*

- *A is a closed subset of M.*
- *A is an open submanifold of M, such that the topological boundary ∂A is a 1-codimensional, orientable, closed submanifold of M.*

The full subcategory of $\mathbf{Man}^2$ *consisting of admissible pairs will be called* **Adm**.

Definition 2.16 *A function $f : M \to \mathbb{R}$ is called a Morse function on an admissible pair (M, A) if one of the following holds:*

- *A is a closed subset of M and $\mathrm{CM}_\bullet(A; f|_A)$ can be identified with a subcomplex of $\mathrm{CM}_\bullet(M; f)$.*
- *A is an open submanifold of M and f is steep with respect to ∂A. This means that $-\nabla f$ is an inner normal vector field of ∂A.*

The latter condition also guarantees that $\mathrm{CM}_\bullet(A; f|_A) \subset \mathrm{CM}_\bullet(M; f)$ can be seen as a subcomplex. This is because no gradient flow lines can leave A, hence the respective boundary operators agree on $\mathrm{CM}_\bullet(A; f|_A)$ by the same argument as for closed embeddings. Now we are in a position to define relative Morse homology.

Definition 2.17 *The* ***relative chain complex*** *of an admissible pair (M, A) with respect to a Morse function $f : M \to \mathbb{R}$ is given by:*

$$\mathrm{CM}_\bullet(M, A; f) := \mathrm{CM}_\bullet(M; f)/\mathrm{CM}_\bullet(A; f|_A)$$

where we consider $\mathrm{CM}_\bullet(A; f|_A)$ to be a subcomplex of $\mathrm{CM}_\bullet(M; f)$ in the obvious manner. The ***relative homology*** *of (M, A) with respect to f is then given by:*

$$\mathrm{HM}_k(M, A; f) = H_k\left(\mathrm{CM}_\bullet(M, A; f)\right)$$

Notice that $\mathrm{CM}_\bullet(M, A; f)$ nicely fits into a short exact sequence of chain complexes:

$$0 \to \mathrm{CM}_\bullet(A; f|_A) \xrightarrow{\iota_\bullet} \mathrm{CM}_\bullet(M; f) \xrightarrow{j_\bullet} \mathrm{CM}_\bullet(M, A; f) \to 0$$

Here $\iota_\bullet$ denotes the inclusion as subchaincomplex and $j_\bullet$ the projection. So by a general fact from homological algebra, we are led to a long exact sequence in homology:

$$\cdots \to \mathrm{HM}_k(A; f|_A) \xrightarrow{(\iota_\bullet)_k} \mathrm{HM}_k(M; f) \xrightarrow{(j_\bullet)_k} \mathrm{HM}_k(M, A; f) \to \mathrm{HM}_{k-1}(A; f|_A) \to \cdots$$

This long exact sequence combined with the algebraic five lemma will allow to transfer most of what we know already about the absolute groups to the relative ones. We will discuss in the next section that these definitions are tailor made to prove the Eilenberg-Steenrod axioms. But first, one needs to redo everything we did so far, to

- Get rid of the dependence on the Morse function.
- Make $\mathrm{HM}_*(\cdot,\cdot)$ into a functor **Adm** $\to$ **Ab**.
- Make sure the relative functor agrees with the non-relative one on pairs of the form $(M,\emptyset)$.

Since it is very similar to the constructions we already did, we will just refer to the book by Schwarz for the details. To get off the ground, we need to exhibit at least one Morse function.

Lemma 2.18 *There is a Morse function on* (M,A).

Proof (Sketch) For closed submanifolds, the extension result in the appendix applies. For open manifolds, one can argue via tubular neighborhood constructions, since ∂A is assumed to be orientable. □

Then one need to make sure that any two choices of Morse functions yield the same homology.

Lemma 2.19 *There is a concept of a relative Morse homotopy such that there is a Morse homotopy h^{rel} between any two Morse functions on an admissible pair (M,A), together with continuation maps $\varphi^{h^{rel}}$. These satisfy all the same properties as the continuation maps in the absolute case.*

Proof (Sketch) The existence of relative Morse homotopies is the content of definition 4.33 and lemma 4.34 in [3]. The relative continuation maps are then obtained by factoring the absolute ones, so that they operate on the quotient complexes. The fact that they induce isomorphisms is then a consequence of the long exact sequence discussed above, the compatibility of the LES with the continuation maps, as well as the algebraic five lemma. This is further explained in $[3, p.186]$. □

This then leads to the independent relative groups via the same limit process as for the absolute groups:

Definition 2.20 *For an admissible pair (M,A), we define the* ***independent relative Morse homology groups*** *by:*

$$\mathrm{HM}_*(M,A) := \varprojlim \mathrm{HM}_*(M,A;f)$$

where the limit is taken over all Morse functions (in the relative sense).

Next, the relative functorial properties have to be established. However this is done exactly the same way as before, first establishing it for diffeomorphisms, then closed embeddings, projections and finally general smooth maps of pairs. Hence we have the following

Theorem 2.21 *The relative Morse homology groups form functors:*

$$\mathrm{HM}_* : \mathbf{Adm} \to \mathbf{Ab}$$

This is proposition 4.38 in [3]. Now finally, it is clear that a relative Morse function on $(M, \emptyset)$ is exactly the same thing as a usual Morse function on M, hence the relative functor is in fact an extension of the absolute one.

2.3 The Axioms

Let's quickly recall all the relevant notions for axiomatic homology theories as established in [5]. First, the category on which we work should allow for some basic operations:

Definition 2.22 *A subcategory* $\mathbf{C}$ *of* $\mathbf{Top}^2$ *is called an* ***admissible category*** *for homology theory if*

- $\mathbf{C}$ *contains all one-point spaces.*
- *For any* $x \in \mathbf{C}$ *and any one-point space* $\{*\}$ *we have* $\hom_{\mathbf{C}}(P, X) = \hom_{\mathbf{Top}^2}(P, X)$.
- *Given a pair* $(X, A) \in \mathbf{C}$*, the following diagram of inclusions (called the* ***lattice*** *of* (X, A)*) is in* $\mathbf{C}$*:*

$$\begin{array}{ccccccccc} & & & & (X,\emptyset) & & & & \\ & & & \nearrow & & \searrow & & & \\ (\emptyset,\emptyset) & \longrightarrow & (A,\emptyset) & & & & (X,A) & \longrightarrow & (X,X) \\ & & & \searrow & & \nearrow & & & \\ & & & & (A,A) & & & & \end{array}$$

  Furthermore, if $f : (X, A) \to (Y, B)$ *is a morphism in* $\mathbf{C}$*, then so are all the maps between members of the lattices of* (X, A) *and* (Y, B) *definable through* f.

- *If* $(X, A) \in \mathbf{C}$*, then so is* $(X \times [0, 1], A \times [0, 1])$*, as well as the maps*

$$\begin{aligned} \iota_i : (X, A) &\to (X \times [0, 1], A \times [0, 1]) \\ (x, a) &\mapsto ((x, t), (a, t)) \end{aligned}$$

  for $i = 0, 1$.

Two maps $f_0, f_1 : (X, A) \to (Y, B)$ are said to be homotopic in an admissible category $\mathbf{C}$ if the homotopy $H : f \equiv g$ is in $\mathbf{C}$. It turns out that $\mathbf{Adm}$ is not an a admissible category, since it only contains manifolds with no boundary. We will however fix this inconvenience by extending Morse homology to a bigger category $\mathbf{Adm} \subset \mathbf{CW}_{reg}$ which is versatile enough to be an admissible category. But first, let's make precise what we mean by the term **homology theory**.

Definition 2.23 *There is a functor $R : \mathbf{Top}^2 \to \mathbf{Top}^2$ such that for*

$$f : (X, A) \to (Y, B)$$

we have

$$R(X, A) = (A, \emptyset) \quad R(f) = f|_A$$

Definition 2.24 *A **homology theory** with coefficient group G on an admissible subcategory $\mathbf{C}$ of $\mathbf{Top}^2$, is a sequence of functors $\mathcal{H}_k : \mathbf{C} \to Ab$ and a sequence of natural transformation $\delta_{k+1} : \mathcal{H}_{k+1} \to \mathcal{H}_k \circ R$ with $k \geq 0$ satisfying the following axioms*

- ***The homotopy axiom**: For any pairs $(X, A), (Y, B) \in \mathbf{C}$, if $f, g : (X, A) \to (Y, B)$ are homotopic in $\mathbf{C}$, then $\mathcal{H}_k(f) = \mathcal{H}_k(g)$ or all $k \geq 0$.*
- ***The long exact sequence axiom**: For any pair $(X, A) \in \mathbf{C}$ with inclusions $i : (A, \emptyset) \hookrightarrow (X, \emptyset)$ and $j : (X, \emptyset) \hookrightarrow (X, A)$, there is a long exact sequence:*

$$\cdots \to \mathcal{H}_k(A) \xrightarrow{\mathcal{H}_k(i)} \mathcal{H}_k(X) \xrightarrow{\mathcal{H}_k(j)} \mathcal{H}_k(X, A) \xrightarrow{\delta_k} \cdots$$

- ***The excision axiom**: For every pair $(X, A) \in \mathbf{C}$ and subset $U \subset X$ such that $\bar{U} \subset A^\circ$ the inclusion $i : (X \setminus U, A \setminus U) \hookrightarrow (X, A)$ induces an isomorphism :*

$$\mathcal{H}_k(X \setminus U, A \setminus U) \cong \mathcal{H}_k(X, A), \quad \forall k \geq 0$$

- ***The dimension axiom**: If $\{*\}$ is a one-point space, then $\mathcal{H}_k(\{*\}) = 0$ for all $k > 0$ and $\mathcal{H}_0(\{*\}) \cong G$*

Let's now finally establish that Morse homology satisfies all of these axioms on **Adm**.

2.3.1 Dimension

Proposition 2.25 *Morse homology satisfies the dimension axiom on* **Adm**. *We have for any one point space $\{*\}$ that $(\{*\}, \emptyset) \in$* **Adm** *and:*

$$\mathrm{HM}_k(\{*\}) = \begin{cases} \mathbb{Z}_2 & , k = 0 \\ 0 & , k > 0 \end{cases}$$

Proof Take the function $f : \{*\} \to \mathbb{R}$ with $f(*) = 0$. It is trivially smooth and , since at its only critical point $*$ the manifold has a 0-dimensional tangent space, it is also trivially Morse. We have, by definition, $\mu(*) = 0$. Hence the Morse chain complex is simply given by

$$\ldots \to 0 \to \ldots \to 0 \to \mathbb{Z}_2 \xrightarrow{0} 0$$

Since all the boundary operators vanish, the result follows. □

2.3.2 Long Exact Sequence

Proposition 2.26 *Morse homology satisfies the long exact sequence axiom on* **Adm**. *For* $(M, A) \in$ **Adm** *there is a long exact sequence:*

$$\cdots \to \mathrm{HM}_k(A, \emptyset) \xrightarrow{\mathrm{HM}_k(\iota)} \mathrm{HM}_k(M, \emptyset) \xrightarrow{\mathrm{HM}_k(j)} \mathrm{HM}_k(M, A) \to \mathrm{HM}_{k-1}(A, \emptyset) \to \cdots$$

where $\iota : (A, \emptyset) \hookrightarrow (M, \emptyset)$ *and* $j : (M, \emptyset) \hookrightarrow (M, A)$ *are inclusions.*

Proof As a consequence of definition 2.17 we have a LES:

$$\cdots \to \mathrm{HM}_k(A; f|_A) \xrightarrow{(\iota_\bullet)_k} \mathrm{HM}_k(M; f) \xrightarrow{(j_\bullet)_k} \mathrm{HM}_k(M, A; f) \to \mathrm{HM}_{k-1}(A; f|_A) \to \cdots$$

This is just a formal consequence of the fact that we defined the relative complex as a quotient complex. One then only needs to verify:

- The LES is compatibel with the continuation maps. More explicitly the homomorphisms $(\iota_\bullet)_k$ and $(j_\bullet)_k$ are compatible with the continuation maps, so that they descend to homology. See Schwarz's book $[3, p.186]$ for details.
- The homomorphisms induced by $(\iota_\bullet)_k$ and $(j_\bullet)_k$ are given by $\mathrm{HM}_*(\iota)$ and $\mathrm{HM}_*(j)$. The case of ι is the content of Lemma 4.37 in Schwarz's book $[3, p.188]$, and j works similarly.
- The resulting sequence on the independent groups is still exact. This is a general property of limits.

So LES axiom holds true for Morse homology. □

2.3.3 Excision

Proposition 2.27 *Morse homology satisfies the excision axiom on* **Adm**. *For* $(M, A) \in$ **Adm**, $B \subset A^o$ *closed and* $(M \setminus B, A \setminus B) \in$ **Adm** *the inclusion* $\iota : (M \setminus B, A \setminus B) \hookrightarrow (M, A)$ *induces an isomorphism:*

$$\mathrm{HM}_*(M \setminus B, A \setminus B) \cong \mathrm{HM}_*(M, A)$$

Proof It is possible to choose a Morse function f on (M, A), so that its restriction to $(M \setminus B, A \setminus B)$ is still a Morse function. Hence the relative chain complexes, which are generated by the critical points of f in $M \setminus A$ and $(M \setminus B) \setminus (A \setminus B) = M \setminus A$ respectively, have isomorphic chain groups in each degree. Using the fact that $B \subset A^o$, it is easy to see that also the boundary operators agree, since flow lines that cross B are not allowed to leave A, and hence do not impact the boundary operator for $\mathrm{CM}_\bullet(M \setminus B, A \setminus B)$. So in fact there is a canonical isomorphism: which in fact induces $\mathrm{HM}_*(\iota; f)$. It is also compatible with the continuation maps, and hence passes to the limit. □

2.3.4 Homotopy

Proposition 2.28 *Morse homology satisfies the homotopy axiom on* **Adm**. *Given two smooth functions* $\chi_0, \chi_1 : (M,A) \to (N,B)$ *and a homotopy* $h_t : \chi_0 \sim \chi_1$, *we have that:*

$$\mathrm{HM}_*(\chi_0) = \mathrm{HM}_*(\chi_1)$$

Proof For the absolute case, this follows from the homotopy lemma. Since $\chi_0 \sim \chi_1$ we also have $(\chi_0, id_M) \sim (\chi_1, id_M)$ through embeddings. Hence the homotopy lemma applies and we get:

$$\mathrm{HM}_*(\chi_0) = p_* \circ (\chi_0, id_M)_* = p_* \circ (\chi_1, id_M) = \mathrm{HM}_*(\chi_1)$$

For the relative case, one needs to reprove the homotopy lemma for maps of pairs and can then proceed in the same manner. □

If we take a moment to reflect on the axioms, we notice that they all followed directly from the properties of Morse homology. This is in stark contrast to other homology theories. In singular homology for instance, excision and homotopy are very non trivial and have long proofs. However, for instance the dimension axiom was easy to verify, since the Morse complex of a one point space was very small and its homology trivial to compute. For compact manifolds, the Morse chain complex is finitely generated (see also proposition 3.2), with simplifies calculations. Next, the LES axiom was bound to be satisfied, since the relative complex was defined as a quotient complex, leading directly to a short exact sequence of chain complexes, which in turn implies the existence of a long exact sequence in homology. The only difficulty lay in the fact that compatibility with the limit process had to be verified and that the morphisms were in fact the right ones. The excision axiom is morally also very easy to accept, since in informal terms it reads: Ignoring the critical points of $f : M \to \mathbb{R}$ inside $A \subset M$ is the same thing as ignoring the critical points of $f|_{M \setminus B} : M \setminus B \to \mathbb{R}$ inside $A \setminus B$. Finally homotopy axioms was based on the homotopy lemma, which in turn followed from the fact that the two inclusions:

$$\iota_i : M \to M \times \mathbb{R}$$

with $\iota_i(p) = (p,i)$ induce the same maps in Morse homology. Now this was, as we already discussed, a consequence the continuation maps relating the two maps before the limit process via :

$$(\iota_0)_* \circ \varphi = (\iota_1)_*$$

so that in the limit they would be identified.

2.4 The Uniqueness Result

We have now concluded the construction of Morse homology as well as shown that it satisfies the Eilenberg-Steenrod axioms for admissible pairs of manifolds. This allows us to exploit the theory developed in [5] to show the main result of this section:

Theorem 2.29 *There is a unique natural isomorphism between singular homology and Morse homology:*

$$\Phi_\bullet : \mathrm{HM}_\bullet \to \mathrm{H}_\bullet^{sing}$$

when seen as homology theories defined on **Adm**.

For the proof of this result we need some additional terminology.

Definition 2.30 *An admissible category* **C** *is called a* ***uniqueness category*** *if for any two homology theories* $(\mathcal{H}_\bullet, \partial)$, $(\mathcal{K}_\bullet, \varepsilon)$ *defined over* **C** *and any homomorphism:*

$$h : \mathcal{H}_0(\{*\}) \to \mathcal{K}_0(\{*\})$$

for a fixed point space, there is a unique natural transformation:

$$\Phi_\bullet : \mathcal{H}_\bullet \to \mathcal{K}_\bullet$$

with $\Phi_0(\{*\}) = h$.

If the two homology theories have isomorphic coefficient groups, the dimension axiom allows h to be an isomorphism, and by uniqueness of $\Phi_\bullet$ this implies that the two ES homology theories are naturally isomorphic. Hence, on a uniqueness category, for any given coefficient group, there is only one homology theory up to natural isomorphism. Here some examples of uniqueness categories

Theorem 2.31 *The following are uniqueness categories :*

- *The category of triangulable spaces (see Theorem 10.4 in* [5, p.120]*)*
- *The category of CW-spaces (see Theorem 46.17 in* [4, l.46]*)*

Unfortunately, the category of admissible pairs is not an admissible category for homology (since it only allows for manifolds without boundary) and hence *a fortiori* not a uniqueness category. This is why we need to extend Morse homology to a bigger category that is admissible.

Definition 2.32 *Let* $\mathbf{CW}_{reg}$ *be the full subcategory of* $\mathbf{CW}^2$ *consisting of all CW-pairs, which can be embedded smoothly into a smooth manifold M, by which is meant that all the characteristic maps of cells are diffeomorphisms relative to their boundary. Furthermore, these pairs must admit open neighborhoods* (U,R) *in M which are admissible and of which they are a strong deformation retract.*

Now this category is admissible, hence it has a chance at being a uniqueness category. Next, we need an appropriate extension of Morse homology onto this bigger category. We will do this with a construction similar to Čech Cohomology.

Definition 2.33 *Given a pair* $M \subset (X,A) \in \mathbf{CW}_{reg}$*, let's consider the family* $\mathfrak{U}_{(X,A)}$ *of open neighborhoods* (U,R) *of* (X,A)*, such that* (X,A) *is a strong deformation retract of* (U,R) *and* (U,R) *is itself admissible. Then this set is directed and preordered under inclusions.*

Definition 2.34 *Given a pair $(X,A) \in \mathbf{CW}_{reg}$, we define:*

$$\check{\mathrm{HM}}_*(X,A) := \varprojlim_{(U,R)\in\mathfrak{U}_{(X,A)}} \mathrm{HM}_*(U,R)$$

as the limit with respect to the inclusion isomorphisms:

$$\iota_{*(V,S)}^{(U,R)} : \mathrm{HM}_*(U,R) \to \mathrm{HM}_*(V,S)$$

As with Čech Cohomology one can make $\check{\mathrm{HM}}$ into a functor. In particular the following holds:

Theorem 2.35 *There is a well defined family of functors:*

$$\check{\mathrm{HM}}_* : \mathbf{CW}_{reg} \to \mathbf{Ab}.$$

Furthermore, $\check{\mathrm{HM}}_\bullet$ *can be given a boundary operator such that* $(\check{\mathrm{HM}}_\bullet, \partial)$ *is an Eilenberg Steenrod homology theory. Finally, there is a natural isomorphism:*

$$\check{\mathrm{HM}}_\bullet \cong \mathrm{HM}_\bullet$$

on $\mathbf{Adm} \subset \mathbf{CW}_{reg}$.

Hence, $\check{\mathrm{HM}}_\bullet$ truly is an extension of Morse homology. Finally we need to show that

Theorem 2.36 $\mathbf{CW}_{reg}$ *is a uniqueness category.*

Proof The proof is the same as for Theorem 46.17 in the notes by Merry [4, *l*.46], which proves that $\mathbf{CW}^2$ is a uniqueness category. Alternatively this is proven in Proposition 4.45 of [3, *p*.196]. The idea in both of these proofs is that for any homology theory $\mathcal{H}_\bullet$ on $\mathbf{CW}_{reg}$, we can construct a natural isomorphism:

$$\mathcal{H}_\bullet \cong \mathcal{H}_\bullet^{cell}$$

between it and an associated cellular homology. Fortunately, the definition $\mathcal{H}_\bullet^{cell}$ can be carried out in $\mathbf{CW}_{reg}$, since it only requires spheres, ball, and wedges of spheres of the same dimension to be contained in it. Merry then goes on to prove that $\mathcal{H}_\bullet^{cell}$ is naturally isomorphic to the usual cellular homology, thereby showing that

$$\mathcal{H}_\bullet \cong \mathcal{H}_\bullet^{cell} \cong \mathrm{H}_\bullet^{cell} \cong \mathcal{K}_\bullet^{cell} \cong \mathcal{K}_\bullet$$

for any other homology theory $\mathcal{K}_\bullet$. Alternatively we can also prove $\mathcal{H}_\bullet^{cell} \cong \mathcal{H}_\bullet^{cell}$ directly using the axioms and the algebraic five lemma. Starting with the homomorphism h to identify 0-spheres in both cellular homologies, one can build up isomorphisms for higher dimensional spheres and balls by Mayer-Vietoris arguments and the LES axiom. Since the Mayer-Vietoris sequence is completely axiomatically deductible, this can be proven for any two $\mathcal{H}_\bullet$ and $\mathcal{K}_\bullet$, hence the result also follows in this case. □

By the previous theorem, $\check{\mathrm{HM}}_\bullet \cong \mathrm{H}_\bullet^{sing}$ on $\mathbf{CW}_{reg}$, and since by the previous lemma, $\check{\mathrm{HM}}_\bullet \cong \mathrm{HM}_\bullet$ on **Adm**. Hence we are now able to conclude that there is a natural equivalence:

$$\mathrm{H}_\bullet^{sing} \cong \mathrm{HM}_\bullet$$

on **Adm**.

Chapter 3

Applications

Now that the isomorphism between Morse homology and singular homology has been established, we can rest assured that all the axiomatically deduceable results which are known from singular homology still hold true for Morse homology. So for instance there is a Mayer-Vietoris sequence, as it's a formal consequence of the long exact sequence axiom and the Barret-Whitehead lemma from homological algebra. However the isomorphism also goes the other way round, so theorems about the Morse homology groups give automatically theorems about the homology groups from any homology theory. In this last chapter, we will present a few results from algebraic topology and examine them in the context of Morse homology.

3.1 Homology of Smooth Manifolds

The homology of smooth, finite-dimensional manifolds is well behaved, as presented by the following results.

Proposition 3.1 *Let M be a smooth m-dimensional manifold that admits a nice pair (f, g). Then:*

$$\mathrm{HM}_k(M) = 0 \text{ if } k < 0 \text{ or } i > m$$

Proof For $p \in \mathrm{Crit}(f)$, we have by definition:

$$0 \leq \mu(p) \leq m$$

since $\mu(p)$ denotes the cardinality of a sub-vectorspace of T_pM. But this implies that

$$k \notin \{0, \cdots, m\} \Rightarrow \mathrm{CM}_k(M; f, g) = 0.$$

Therefore the homology $\mathrm{HM}_k(M; f, g)$ must do the same. Since $\mathrm{HM}_k(M) \cong \mathrm{HM}(M; f, g)$ the claim follows. □

In particular, we know from section 1.5 that the result holds true for $\mathbb{R}^n$, closed manifolds and vector bundles over closed manifolds. For closed manifold we can say even more:

Proposition 3.2 *Let M be a closed m-dimensional manifold. Then all homology groups $\mathrm{H}_k(M)$ are finitely generated.*

Proof Let $f : M \to \mathbb{R}$ be Morse and choose the Riemannian metric g such that (f, g) is a nice pair. Then by proposition 1.6 and the subsequent remark, we know that $|\mathrm{Crit}(f)| < \infty$. Hence all the groups of the Morse complex $\mathrm{CM}_\bullet(M; f, g)$ are finitely generated, and so are the homology groups. The claim then follows from $\mathrm{HM}_k(M) \cong \mathrm{HM}(M; f, g)$. □

Given only singular homology, these facts are difficult to deduce as the singular chain complex is very large, however in Morse homology, which is cellular homology in disguise, to come very easily. Morse homology is however more versatile than cellular homology, since we can choose the Morse function to fit our specific needs, and adjust the Morse cell complex accordingly. An example of this is the following result concerning the top- and bottom-dimensional homology group of a closed manifold.

Proposition 3.3 *Let M be a connected smooth closed m-dimensional manifold. Then*

$$\mathrm{HM}_0(M) = \mathrm{HM}_m(M) = \mathbb{Z}_2.$$

Proof (Sketch) If we can find a Morse function $f : M \to \mathbb{R}$ such that f has exactly one minimum respectively maximum, then choosing a nice metric, the corresponding homology will satisfy $\mathrm{HM}_0(M; f, g) = \mathbb{Z}_2$ respectively $\mathrm{HM}_k(M; f, g) = \mathbb{Z}_2$. So it remains to show the existence of such f. There are procedures that start with any Morse function, and yield one that has only 1 minimum. It works by carefully connecting minima via flow lines through index 1 critical points into a tree, and then homotoping f so that all the minima converge together along the flow lines. Applying this procedure to $-f$ and then negating the output yields a function with exactly one maximum. Hence we're done. See also [6]. □

3.2 Morse Cohomology

Given a homology theory based on a chain complex, one can apply the $\hom(\square, A)$ functor for some abelian group A to obtain a cochain complex and a corresponding cohomology theory with coefficients in A. This is for instance the relationship between singular homology and singular cohomology. Exactly the same procedure can be applied to Morse homology to obtain the corresponding cohomology theory. However, we will need to be a little more careful, since Morse homology is in general obtained from a infinite number of chain complexes. In other words, we have to make sure that the continuation maps carry over to the cochain complexes. Since we're always working with $\mathbb{Z}_2$-coefficients, we will suppress the coefficient group from the notation.

Definition 3.4 *Let (f, g) be a nice pair on an m-dimensional smooth manifold M. We define their **Morse ($\mathbb{Z}_2$-)co-chain complex***

$$(\mathrm{CM}^\bullet(M; f, g), d)$$

to be given by the following data:

- *For all $k \in \mathbb{Z}$ let* $\mathrm{CM}^k(M; f, g) = \hom(\mathrm{CM}_k(M; f, g), \mathbb{Z}_2)$.
- *Let $\alpha \in \mathrm{CM}^k(M; f, g)$. Define the coboundary operator by*

$$(d\alpha)(x) := \alpha(\partial x)$$

for all $x \in \mathrm{Crit}_k(f) \subset \mathrm{CM}_k(M; f, g)$ and extend by linearity.

Now that we have a cochain complex we can take its cohomology :

Definition 3.5 *Given a nice pair (f, g) on a manifold M, their* ***Morse cohomology*** *with $\mathbb{Z}_2$ is defined as :*

$$\mathrm{HM}^k(M; f, g) = \mathrm{H}^k(\mathrm{CM}^\bullet(M; f, g))$$

Recall the universal coefficients theorem for $\mathbb{Z}_2$-coefficients. Given a chain complex $C_\bullet$ of free $\mathbb{Z}_2$-modules, for every $n \geq 0$ there is a split exact sequence sequence :

$$0 \to \mathrm{Ext}_1^{\mathbb{Z}_2}(\mathrm{H}_{n-1}(C_\bullet), \mathbb{Z}_2) \to \mathrm{H}^n(\hom(C_\bullet, \mathbb{Z}_2)) \xrightarrow{\zeta} \hom(\mathrm{H}_n(C_\bullet), \mathbb{Z}_2) \to 0$$

where $\zeta\langle\gamma\rangle\langle c\rangle = \gamma(c)$. Here $\gamma \in \mathrm{H}^n(\hom(C_\bullet, \mathbb{Z}_2))$ and $c \in \mathrm{H}_n(C_\bullet)$. Using the fact that $\mathrm{Ext}_1^{\mathbb{Z}_2}(V, \mathbb{Z}_2) = 0$ for any $\mathbb{Z}_2$-vector space V, we get that:

$$\mathrm{H}^n(\hom(C_\bullet, \mathbb{Z}_2)) \cong \hom(\mathrm{H}_n(C_\bullet), \mathbb{Z}_2)$$

respectively in Morse homology terms:

$$\mathrm{HM}^n(M; f, g) \cong \hom(\mathrm{HM}_n(M; f, g), \mathbb{Z}_2).$$

This allows us to define a Morse cohomology independent of the choice of a nice pair.

Proposition 3.6 ***Morse ($\mathbb{Z}_2$)-cohomology*** *of a closed manifold M defined analogously to Morse homology by:*

$$\mathrm{HM}^k(M) := \varinjlim_{(f,g)\in NP} \mathrm{HM}^k(M; f, g)$$

is well defined.

Proof Consider the continuation map:

$$\Phi^{\alpha\beta} : \mathrm{CM}_\bullet(M; f^\beta) \to \mathrm{CM}_\bullet(M; f^\alpha)$$

which induces an isomorphism $\varphi_k^{\alpha\beta}$ in Morse homology. Here we suppress the metric from the notation as it is irrelevant at the moment. Consider the following diagram:

$$\begin{array}{ccc}
\mathrm{HM}^k(M; f^\alpha) & \xrightarrow{\mathrm{H}^k(\hom(\Phi, \mathbb{Z}_2))} & \mathrm{HM}^k(M; f^\beta) \\
\downarrow \zeta & & \downarrow \zeta \\
\hom(\mathrm{HM}_k(M; f^\alpha), \mathbb{Z}_2) & \xrightarrow{\hom(\mathrm{H}^k(\Phi), \mathbb{Z}_2)} & \hom(\mathrm{HM}_k(M; f^\beta), \mathbb{Z}_2)
\end{array}$$

where the vertical arrows given by ζ are isomorphisms. It then follows by commutativity that

$$\psi^{\alpha\beta} = \mathrm{H}^k(\hom(\Phi^{\alpha\beta}, \mathbb{Z}_2))$$

form a set of continuation maps cohomology with the same properties as the $\varphi^{\alpha\beta}$. Explicitely:

- $\psi^{\alpha\beta}$ is an isomorphism
- $\psi^{\alpha\beta} \circ \psi^{\beta\gamma} = \psi^{\alpha\gamma}$
- $\psi^{\alpha\alpha} = \mathrm{id}$

Now we can proceed exactly as for homology to conclude that the limit exists and is isomorphic to $\mathrm{HM}^k(M; f)$ for any nice pair (f, g). □

Next, functorial behaviour can also be transferred via the UCT. Hence HM^k can be made into a functor. In fact, the following holds:

Theorem 3.7 *Morse cohomology can be extended to a cohomology theory in the sense of Eilenberg and Steenrod.*

Proof (Sketch) We can consider a relative cochain complex :

$$\mathrm{CM}^\bullet(M, A; f, g) = \hom(\mathrm{CM}_\bullet(M; f, g)/\mathrm{CM}_\bullet(A; f_A, g_A), \mathbb{Z}_2)$$

and define $\mathrm{HM}^k(M, A; f, g) = \mathrm{H}^k(\mathrm{CM}^\bullet(M, A; f, g))$. Then via the universal coefficient theorem it can be shown that these groups are well behaved under taking limits, hence yield groups independent of the nice pair. Finally, one goes trough the list of axioms, applies hom to them in a suitable sense and uses the UCT once again to see that they carry over to cohomology. □

The analog in cohomology of the uniqueness theorem for Eilenberg-Steenrod homology theories then tells us that:

$$\mathrm{HM}^k \cong \mathrm{H}^k_{sing}$$

Hence, a theorem on Morse cohomology groups yields an equivalent theorem for any other ES-cohomology theory.

3.3 Poincaré Duality

In this section, we will exploit the correspondance between Morse functions $f : M \to \mathbb{R}$ and their opposite $-f$. If M is compact, then this is again a Morse function. This is the crux on which the famous duality theorem rests.

Theorem 3.8 (Poincaré duality) *Let M be a closed m-dimensional manifold. Then:*

$$\mathrm{HM}_k(M) \cong \mathrm{HM}^{m-k}(M)$$

for any $k \in \mathbb{N}$.

Proof Consider a Morse function $f : M \to \mathbb{R}$. As M is compact, $-f$ is again a coercive function, hence also an admissible function for Morse homology. By theorem 1.37 there is a continuation map:

$$\Phi_\bullet : \mathrm{CM}_\bullet(M; f) \to \mathrm{CM}_\bullet(M; -f)$$

in particular Φ is a chain equivalence, and hence induces isomorphisms $\varphi_k = \mathrm{HM}_k(\Phi_\bullet)$ in homology. Next, we clearly have $\mathrm{Crit}(f) = \mathrm{Crit}(-f)$. More precisely, $\mathrm{Crit}_k(f) = \mathrm{Crit}_{m-k}(-f)$, since by changing the sign of f, the eigenvalues of the hessian at any point also get their sign changed. Hence we have an isomorphism:

$$\begin{aligned} K_k : \mathrm{CM}_k(M; -f) &\to \mathrm{CM}_{m-k}(M; f) \\ x &\mapsto x \end{aligned}$$

Finally, since by the proof of proposition 3.2, $\mathrm{CM}_k(M; f)$ is a finitely generated $\mathbb{Z}_2$-vector space with a chosen basis given by the critical points, there is canonical isomorphism:

$$\begin{aligned} \Gamma_k : \mathrm{CM}_{m-k}(M; f) &\to \mathrm{CM}^{m-k}(M; f) \\ x &\mapsto \delta_x \end{aligned}$$

where $\{\delta_x\}$ is the basis dual to $\mathrm{Crit}_{m-k}(f)$. Combining the last two assertions gives us a family of isomorphisms:

$$\Psi_k = \Gamma_k \circ K_k : \mathrm{CM}_k(M; -f) \to \mathrm{CM}^{m-k}(M; f)$$

which is in fact can be assembled into a chain maps, since the following commutes:

$$\begin{array}{ccc} \mathrm{CM}_k(M; -f) & \xrightarrow{\partial} & \mathrm{CM}_{k-1}(M; -f) \\ \downarrow{\scriptstyle \Psi_k} & & \downarrow{\scriptstyle \Psi_{k-1}} \\ \mathrm{CM}^{m-k}(M; f) & \xrightarrow{d} & \mathrm{CM}^{m-k+1}(M; f) \end{array}$$

This is because for $x \in \mathrm{Crit}_k(-f)$ and $z \in \mathrm{Crit}_{m-k+1}(f)$:

$$\begin{aligned} \Psi_{k-1}(\partial_{-f} x)(z) &= \sum_{y \in \mathrm{Crit}_{k-1}(-f)} [x, y]^{-f} \delta_y(z) = [x, z]^{-f} \\ d\Psi_k(x)(z) &= \sum_{y \in \mathrm{Crit}_{m-k}(f)} [z, y]^f \delta_x(y) = [z, x]^f \end{aligned}$$

where wee need to keep in mind that $\mathrm{CM}_k(M; f) = \mathrm{CM}_{m-k}(-f)$. Now, there are clearly as many flow lines of f going from z to x as there are flow lines of $-f$ going from x to z. Hence $[x, z]^{-f} = [z, x]^f$, and the above diagram commutes. Since we are now dealing with a chain map, we get further isomorphisms $\psi_k = \mathrm{HM}_k(\Psi_\bullet)$ in homology. Combining everything, we are left with isomorphisms:

$$PD_* = \psi_* \circ \varphi_* : \mathrm{HM}_*(M; f) \to \mathrm{HM}^{m-*}(M; f)$$

which is the desired Poincaré duality isomorphism. □

3.4 Künneth Formula

Finally, we'll look at the homology of a product manifold. Remember that if f_M, f_N are Morse functions on M and N respectively, a natural choice of Morse function on $M \times N$ is $f_M \oplus f_N$. This addivity makes it easy to write down an explicit Eilenberg-Zilber morphism:

Proposition 3.9 *Given two closed manifolds M and N together with nice pairs* $(f_M, g_M) \in \mathbf{NP}(M)$ *and* $(f_N, g_N) \in \mathbf{NP}(N)$. *Then there is an isomorphism :*

$$\kappa_\bullet : \mathrm{CM}_\bullet(M \times N; f_M \oplus f_N) \to \mathrm{CM}_\bullet(M; f_M) \otimes_{\mathbb{Z}_2} \mathrm{CM}_\bullet(N; f_N)$$

where $\kappa_n(x_i, y_{n-i}) = x_i \otimes_{\mathbb{Z}_2} y_{n-i}$.

Proof First we need to make sure $\kappa_\bullet$ is a chain map. Consider:

$$\begin{aligned}
\kappa_\bullet \partial(x,y) &= \sum_{(u,v)\in \mathrm{Crit}(f_M \oplus f_N)} [(x,y),(u,v)]x \otimes_{\mathbb{Z}_2} y \\
&= \sum_{u \in \mathrm{Crit}(f_M)} [(x,y),(u,y)]u \otimes_{\mathbb{Z}_2} y + \sum_{v \in \mathrm{Crit}(f_N)} [(x,y),(x,v)]x \otimes_{\mathbb{Z}_2} v \\
&= \left(\sum_{u \in \mathrm{Crit}(f_M)} [x,u]u \right) \otimes_{\mathbb{Z}_2} y + x \otimes_{\mathbb{Z}_2} \left(\sum_{v \in \mathrm{Crit}(f_N)} [y,v]v \right) \\
&= (\partial x) \otimes_{\mathbb{Z}_2} y + x \otimes_{\mathbb{Z}_2} (\partial y) \\
&= \partial(x \otimes_{\mathbb{Z}_2} y) = \partial \kappa_\bullet(x,y)
\end{aligned}$$

Using now the fact that

$$\mathrm{Crit}_n(f_M \oplus f_N) = \bigcup_{l+k=n} \mathrm{Crit}_l(f_M) \times \mathrm{Crit}_k(f_N)$$

as well as properties of the tensor product, we see that $\mathrm{CM}_k(M \times N; f_M \oplus f_N)$ and $(\mathrm{CM}_\bullet(M; f_M) \otimes_{\mathbb{Z}_2} \mathrm{CM}_\bullet(N; f_N))_k$ are free $\mathbb{Z}_2$−modules of the same finite dimension. Furthermore, κ_k is surjective, hence also injective. From this it follows that $\kappa_\bullet$ is indeed an isomorphism. □

On the other hand we have the purely homological result, known as the algebraic Künneth formula, which relates the homology of a tensor complex to the homologies of the factors.

Theorem 3.10 (Algebraic Künneth formula) *Let R be a principal ideal domain. For any two chain complexes* $(C_\bullet, \partial), (D_\bullet, \partial)$ *of R-modules and any* $n \in \mathbb{N}$ *there is a split exact sequence:*

$$0 \to \bigoplus_{i+j=n} \mathrm{H}_i(C_\bullet) \otimes_R \mathrm{H}_j(D_\bullet) \xrightarrow{\omega_n} \mathrm{H}_n(C_\bullet \otimes_R D_\bullet) \to \bigoplus_{l+k=n-1} \mathrm{Tor}_1^R(\mathrm{H}_i(C_\bullet), \mathrm{H}_j(D_\bullet)) \to 0$$

where $\omega_n(\langle c_i \rangle \otimes_R \langle d_{n-i} \rangle) = \langle c_i \otimes_R d_{n-i} \rangle$.

In our special case, where $R = \mathbb{Z}_2$ and all the $\mathrm{H}_i(C_\bullet)$ are vector spaces over $\mathbb{Z}_2$, we get the simplified statement that there is an isomorphism ω_n:

$$\mathrm{H}_n(C_\bullet \otimes_{\mathbb{Z}_2} D_\bullet) \cong \bigoplus_{i+j=n} \mathrm{H}_i(C_\bullet) \otimes_{\mathbb{Z}_2} \mathrm{H}_j(D_\bullet)$$

If we now combine these two results, we get the topological Künneth formula for Morse homology

Proposition 3.11 *Let M and N be two closed manifolds, together with nice pairs $(f_M, g_M) \in \mathbf{NP}(M)$ and $(f_N, g_N) \in \mathbf{NP}(N)$. Then there is an isomorphism:*

$$\kappa_n : \mathrm{HM}_n(M \times N; f_M \oplus f_N, g_M \oplus g_N) \to \bigoplus_{i+j=n} \mathrm{HM}_i(M; f_M, g_M) \otimes_{\mathbb{Z}_2} \mathrm{HM}_j(N; f_N, g_N)$$

given by: $\kappa_n(\langle (x_i, y_j) \rangle) = \langle x_i \rangle \otimes_{\mathbb{Z}_2} \langle y_j \rangle$

Now by the usual argument about invariance under homotoping Morse functions we can end the thesis on the following result for the independent groups:

Theorem 3.12 (Künneth formula) *Let M and N be two closed manifolds. Then there is an isomorphism:*

$$\kappa_n : \mathrm{HM}_n(M \times N) \to \bigoplus_{i+j=n} \mathrm{HM}_i(M) \otimes_{\mathbb{Z}_2} \mathrm{HM}_j(N)$$

induced by the isomorphism of the previous theorem.

Appendix A

Extension of Morse functions

Let $N \subset M$ be two smooth manifolds with M closed. Sometimes, it is useful to be able to extend Morse functions $f \in C^\infty(N, \mathbb{R})$ to $g \in C^\infty(M, \mathbb{R})$ on the whole manifold. Ideally in a way such that the critical points get maintained as well as their index, i.e. $\mathrm{Crit}_k(f) \subset \mathrm{Crit}_k(g)$ for all $k \in \mathbb{N}$. If N is an open subset this is relatively easy, however if N has nonzero codimension we have to be a little careful not to alter the index.

First a lemma that lets us extend a Morse function from a closed subset, if we are already given an extension to an open neighborhood. In the proof we will make use of the strong or Whitney topology on $C^k(M, \mathbb{R})$. A basis of this topology is given by the open sets

$$\mathcal{N}_\varepsilon^k(f) = \{g \in C^k(M, \mathbb{R}) : \forall x \in M\ \forall |s| \leq k |\partial_s g(x) - \partial_s f(x)| < \varepsilon_s(x)\}$$

for all $f, \varepsilon_s \in C^k(M, \mathbb{R})$, s being a multi-index and ∂_s the corresponding differential operator. For non-compact M, this is a stronger topology than the one induced by the $W^{k,\infty}$-norm, where only constant ε_s are considered. For M compact, they in fact agree, since ε_s must attain a non-zero minimum. $C^k(M, \mathbb{R})$ endowed with this topology will be denoted by $C^k_s(M, \mathbb{R})$.

Lemma A.1 *Let M be smooth and closed, $A \subset M$ a closed set, $W \subset M$ an open set and $f \in C^\infty(W, \mathbb{R})$ Morse, such that:*

$$\mathrm{Crit}(f) \subset A^\circ \subset A \subset W \subset M.$$

Then there is $g \in C^\infty(M, \mathbb{R})$ Morse and coercive, such that $g|_A = f|_A$.

Proof Since M is normal, we can find an open set U such that $A \subset U \subset \bar{U} \subset W$. Furthermore, normality allows us to apply Tietze's extension theorem from topology. We get:

$$g_0 \in C^0(M, \mathbb{R}) \text{ such that } g_0|_{\bar{U}} = f|_{\bar{U}}$$

Choose an open convex neighborhood $\mathcal{N}_\varepsilon^2(g_0) \subset C_s^2(\bar{U}, \mathbb{R})$ such that for all $h \in \mathcal{N}_\varepsilon^2(g_0)$ we have $\mathrm{Crit}(h) \cap (U \setminus A) = \emptyset$, i.e. restricting to U, all the critical points of h lie in A. This can be achieved, since g_0 has no critical points in this region, and we can explicitely bound away from 0 the derivatives of h by choosing appropriate ε_s. A short note, this would not be possible in the standard topology, we do need the freedom to choose our open sets with precision in this case. Let then

$$\mathcal{N} = \{f \subset C_s^2(M, \mathbb{R}) : f|_{\bar{U}} \in \mathcal{N}_\varepsilon^2(g_0)\},$$

which is also a convex open neighborhood. Consider a map $g \in C^\infty(U, \mathbb{R})$ where $U \subset \mathbb{R}^n$ is open. g is Morse exactly when

$$G_U(x) = (\det(M_x(g)))^2 + \sum_{k=1}^{n} |\partial_k g(x)|^2 > 0$$

for all $x \in U$. If we now consider $g \in C^\infty(M, \mathbb{R})$ on a closed manifold M, where M has a finite atlas $\{(U_i, \varphi_i)\}_{i=1}^N$, then a similar statement holds true. Let ρ_i be a partition of unity subordinate to $\{U_i\}$. Then g is Morse if and only if

$$\sum_{i=1}^{N} \rho_i G_{U_i}(\varphi_i(p)) > 0$$

From this it is clear, that $X := \{g \in C^2(M, \mathbb{R}) : g$ has only nondegenerate critical points$\}$ is an open subset of $C_s^2(M, \mathbb{R})$, again by bounding the first and second derivatives appropriately away from 0. Since X as well as $C^\infty(M, \mathbb{R})$ lie dense in $C_s^2(M, \mathbb{R})$, we can find

$$g_1 \in C^\infty(M, \mathbb{R}) \cap \mathcal{N} \cap X$$

arbitrarily close to g_0 in $C_s^2(M, \mathbb{R})$. Consider a cutoff-function: $\alpha : M \to [0, 1]$ such that:

$$\alpha|_A = 1 \text{ and } \alpha|_{MU} = 0$$

This allows us to define:

$$g := \alpha g_0 + (1 - \alpha) g_1$$

One readily checks, that $g \in C^\infty(M, \mathbb{R}) \cap \mathcal{N} \cap X$ by convexity of the neighborhoods and the choice of α, given that g_0 and g_1 are close enough to each other, so that no additional critical points get created. Furthermore

$$g|_A = f|_A$$

again by the choice of α. □

Now we are ready to prove the following extension result:

Proposition A.2 *Given a closed submanifold $N \subset M$ of a closed Riemmanian manifold (M, g), as well as a Morse function $f_N \in C^\infty(N, \mathbb{R})$, there is a Morse function $f \in C^\infty(M, \mathbb{R})$ extending f_N:*

$$f|_N = f_N$$

such that $\mathrm{Crit}_k(f_N) \subset \mathrm{Crit}_k(f)$.

Proof Consider the normal bundle of N denoted by

$$TN^{\perp} = \{(p, v) \in TM : g_p(v, T_pN) = 0\}$$

One version of the tubular neighborhood theorem states that $TN^{\perp}$ can be embedded diffeomorphically into M. More precisely, we know that there is an open neighborhood W of N and a diffeomorphism $\varphi : TN^{\perp} \xrightarrow{\cong} W$ such that the following commutes:

$$\begin{array}{ccc} TN^{\perp} & \xrightarrow{\varphi} & W \subset M \\ \pi \downarrow \uparrow & \nearrow & \\ N & & \end{array}$$

where the upward pointing arrows are the obvious inclusions and $\pi : TN^{\perp} \to N$ is the projection of $TN^{\perp}$ seen as a vector bundle over N. Let the metric on the normal bundle be the appropriate restriction of g. Now consider the through f_N and g induced Morse function $f_{TN^{\perp}}$ as defined in chapter 1. Then we get a Morse function $f_W \in C^{\infty}(W, \mathbb{R})$ via

$$f_W = f_{TN^{\perp}} \circ \varphi^{-1}$$

with $\mathrm{Crit}(f_W) \subset N$. Define further for $R > 0$

$$A_R := \varphi(TN^{\perp}_R)$$

where $TN^{\perp}_R := \{(p, v) \in TN^{\perp} : g_p(v, v) \leq R\}$. This clearly is a closed subset of M such that :

$$\mathrm{Crit}(f_W) \subset N \subset A_1^{\circ} \subset A_1 \subset W \subset M,$$

hence we may apply the previous lemma, to conclude that there is a Morse function $f \in C^{\infty}(M, \mathbb{R})$ with $f|_N = f_N$. The fact that the index of the critical points are maintained follows from the definition of a Morse function on a vector bundle. It locally has the form

$$f_{TN^{\perp}}|_U = f \oplus q$$

with some positive definite quadratic form q. Now by additivity of the index and the fact that the index of the only critical point of q is 0 (as it is a local minimum), the claim follows. □

We can even choose the extension in such a way that no matter what the underlying metric on M, flow lines starting in N stay in N.

Corollary A.3 *Let g be a metric on M, such that (f, g) is nice, where f is the extension of f_N constructed above. Then no flow line of f leaves N.*

Proof Due to the construction of $f_{TN^{\perp}}$ via the Riemmanian metric on $TM^{\perp}$, we have that for any $\varepsilon > 0$, $-\nabla_{TN^{\perp}} f_{TN^{\perp}}$ points inward on the boundary of the manifold with boundary $TN^{\perp}_R$. Hence $-\nabla_W f_W$ also points inward on ∂A_R for any $0 < R < 1$. So a trajectory starting in N cannot stray any positive distance away from N. □

The results from this section can even be extended to non compact manifolds:

Theorem A.4 *Given a closed submanifold $N \subset M$ of a Riemmanian manifold (M, g), as well as a coercive Morse function $f_N \in C^\infty(N, \mathbb{R})$, there is a coercive Morse function $f \in C^\infty(M, \mathbb{R})$ extending f_N:*

$$f|_N = f_N$$

such that $\mathrm{Crit}_k(f_N) \subset \mathrm{Crit}_k(f)$.

The proof is more involved, but similar in spirit as the one given here. Notice that we require all Morse functions to be coercive, since this a key requirement for the definition of Morse homology for non compact manifolds.

Bibliography

[1] Augustin Banyaga and David Hurtubise. *Lectures on Morse Homology*. Kluwer Academic Press, 2004.

[2] John Milnor. *Morse Theory*. Princeton University Press, 1960.

[3] Matthias Schwarz. *Morse Homology*. Springer, 1993.

[4] Will J. Merry. Lecture notes on Algebraic Topology, August 2018.

[5] Samuel Eilenberg and Norman Earl Steenrod. *Foundations for Algebraic Topology*. Princeton University Press, 1952.

[6] Morse function with exactly one maximum. https://math.stackexchange.com/questions/163741/any-manifold-admits-a-morse-function-with-one-minimum-and-one-maximum. 28.08.2018.

List of Symbols

M, N, P	names of paracompact Hausdorff manifolds
TM	tangent bundle to the manifold M
T_pM	tangent space at the point p to the manifold M
$TM^\star$	cotangent bundle to the manifold M
$T_pM^\star$	cotangent space at the point p to the manifold M
df_pv	differential of f at the point p applied to v
$\mathrm{Crit}(f)$	critical points of f
$\mathrm{Crit}_k(f)$	critical points of index k of f
$H_p(f)$	hessian of f at the point p
$M_p(f)$	hessian matrix of f at the point p w.r. to some parametrization ψ
$\mu(x)$	index of a critical point x
$\Re(M)$	set of Riemmanian metric on M together with C^∞ topology
$M^f_{x,y}$	trajectory space between x and y of f
$\hat{M}^f_{x,y}$	reduced trajectory space
$M^{h^{\alpha\beta}}_{x,y}$	homotopy trajectory space between x and y
$M^{H^{\alpha\beta}}_{x,y}$	λ-trajectory space
$\mathrm{CM}_\bullet(M; f, g)$	Morse complex
$(\mathrm{CM}_\bullet, \partial)_{f,g}$	Morse complex with emphasis on boundary operator
$\mathrm{HM}_k(M; f, g)$	k-th $\mathbb{Z}_2$-Morse homology group
$\mathrm{HM}_k(M)$	k-th independent $\mathbb{Z}_2$-Morse homology group
$\Phi^{\alpha\beta}$	chain maps between $\mathrm{CM}_\bullet(M; f^\alpha, g^\alpha)$ and $\mathrm{CM}_\bullet(M; f^\beta, g^\beta)$
$\varphi^{\alpha\beta}_k$	induced maps in degree k homology of the $\Phi^{\alpha\beta}$
$\Psi^{\alpha\beta\gamma}$	chain homotopies between $\Phi^{\beta\gamma} \circ \Phi^{\alpha\beta}$ and $\Phi^{\alpha\gamma}$