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The Model of Solovay

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Abstract

It is a well known fact that under **AC** not every set of reals is Lebesgue measurable. However the Model of Solovay demonstrates that when the weaker **DC** is assumed instead, all subsets of the real line may be measurable. We construct the Model of Solovay, show that its reals line satisfies further regularity properties, and demonstrate the interplay between the regularity of the reals and different alternatives to **AC**.

Prelude

The objective of measure theory is to make rigorous the intuitive notions of length, surface are, volume, and measurement of quantity. A mathematical model of all of these can be found in the concept of *measure* on a set S, which in its simplest form is a function $\mu : 2^S \to [0, 1]$ such that the following hold:

(i) Monotonicity: $X \subset Y \Rightarrow \mu(X) \leq \mu(Y)$

(ii) σ-additivity: $\mu(\sqcup_{n\in\omega} X_n) = \Sigma_{n\in\omega}\mu(X_n)$

(iii) Normalization: $\mu(\emptyset) = 0$ and $\mu(S) = 1$

(iv) Atomlessness: $\forall a \in S : \mu(\{a\}) = 0$

On the one hand, this notion replicates a number of facts about volume in the physical world. For instance, the first condition can be rephrased as "Bigger objects have bigger volume" and the second one translates to "The combined volume of a collection of disjoint objects is the sum of their volumes".

On the other hand, we can interpret a measure as an order-preserving map between partially ordered sets. Here the set 2^S is ordered by inclusion, and $\mathbb{R}$ is ordered by $\leq$. From this perspective, (i) just means that μ is order-preserving, (ii) restricts the value of μ on certain joins, and (iii) and (iv) are normalization conditions. In analogy of the representation theory of groups, the structure of 2^S as a poset is compared trough μ to the poset structure of $[0, 1]$. This interrelates the two structures and allows to analyze the structure of either one.

With this definition in mind, there are a few questions that arise naturally. From the practical point of view, me might ask:

Question *Does euclidean space $\mathbb{R}^n$ admit a measure that is translation invariant and hence reflects our experience of volume?*

From the set theoretic point of view, one might ask:

Question (Measure Problem) *Which sets S admit a measure? In particular, since S admits a measure exactly when $|S|$ does, which cardinals admit measures?*

As we will show later, the answer to the first question is no under **AC**. This has peculiar consequences such as the possibility to decompose a solid ball into a small number of pieces, rearranging them, and ending up with two exact copies of the initial ball. This mathematical possibility of copying a ball without adding any mass is known as the *Banach-Tarski paradox*, and would have been impossible if a measure existed on $\mathbb{R}^3$, for then mass would have to be conserved.

There are essentially two possible directions to proceed. Either one can weaken the requirement that the measure be translation invariant, in which case there may exist a solution (see chapter 10 of [1] for an exposition on general measures on sets), or one can weaken **AC**, which is the route we are taking.

Robert M. Solovay showed in 1970 (see [2]) that there is a model of set theory in which every set of reals is Lebesgue measurable and where a such a weakening of **AC**, the axiom of dependent choice **DC**, still holds. The main goal of this thesis is to construct this model and to discuss how the structure of the real line relates to strong assumptions, such as further axioms of set theory or large cardinal hypotheses.

In the first chapter, we introduce essential concepts and tools from descriptive set theory, such as a representation of the reals used in set theory, the Borel sets and alongside Lebesgue measurability two other regularity properties of the reals, the property of Baire and the perfect set property.

The second chapter constructs the Model of Solovay and proves that any subset of the real line in it is Lebesgue measurable, has the Property of Baire and the perfect set property. The proof requires the existence of an inaccessible cardinal to define the Lévy collapse, which is a forcing notion that allows to overwrite any bad behaviour the reals might display. With this a possible solution to the measure problem is presented if **AC** is not assumed.

In the last chapter we present the work of Shelah and Specker, illuminating when the assumption of the inaccessible cardinal is really needed, and in which cases it can be avoided. Finally, we will compare different alternatives to **AC** and present their impact on the regularity of the reals.

We generally follow the books by Jech [1] and Kanamori [3].

I would like to express my sincere thanks to PD Dr. Lorenz Halbeisen for introducing me to the world of set theory as well as his excellent guidance throughout the preparation of this thesis.

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Chapter 1

Descriptive Set Theory

Descriptive set theory studies the properties of the real line. Questions about general subsets of $\mathbb{R}$ may be very difficult to answer, but become approachable when attention is restricted to a collection of ”uncomplicated” subsets. In this chapter we will study the Borel sets, a definable ”skeleton” of the subsets of the real line with a number of topological and measure theoretic properties. Then we will discuss three regularity properties of the real line, which define further ”uncomplicated” sets. But first we will explore a more handy representation of the reals for set theoretic purposes.

1.1 The Baire Space

Definition 1.1 *The Baire space $\mathcal{N}$ is the topological space of countably infinite sequences in ω, denoted as a set by $\omega^\omega = \{\langle x_i : i \in \omega\rangle : x_i \in \omega\}$, and the product topology. Here each ω should be fitted with the discrete topology, in which every set is open.*

A basis of the topology of $\mathcal{N}$ can be described explicitly. Consider a finite sequence with of natural numbers $s \in \omega^{<\omega} = \bigcup_{n\in\omega} \omega^n$. In other words let $s = \langle s_i : i < n\rangle$ for some $n \in \omega$. Define:

$$O(s) = \{\langle x_i : i \in \omega\rangle_{i\in\omega} : \forall i < n\ (x_i = s_i)\} \subset \mathcal{N}$$

Then the $O(s)$ for $s \in \omega^{<\omega}$ form a basis of the topology of $\mathcal{N}$. The Baire space is a prominent example of a larger class of spaces, the *Polish spaces*. They are so named since they were first studied extensively by Polish mathematicians.

Definition 1.2 *A Polish space is a topological space that is homeomorphic to a complete seperable metric space.*

Metrizability of the Baire space is not an issue, as one can easily check that:

$$d\left(\langle x_i\rangle_{i\in\omega}, \langle y_i\rangle_{i\in\omega}\right) = \begin{cases} 0, & \text{if } \forall i \in \omega(x_i = y_i) \\ \frac{1}{\min\{i\in\omega : x_i \neq y_i\}+1}, & \text{else} \end{cases}$$

defines a metric on $\mathcal{N}$ which induces the product topology. Convergence in $\mathcal{N}$ with respect to this metric means being finally constant in each component, hence the Baire space together with d is complete. As $\mathcal{N}$ admits a countable basis to its topology, it is also seperable as a metric space. Hence the Baire space is an example of a Polish space. Other examples include the real line as well as the Hilbert cube.

The Baire space however is special among Polish spaces for two reasons. First, it is in fact the prototypical example of a Polish space as explained by the following classical theorem:

Theorem 1.1 *Every Polish space is the image of the Baire space under a continous map.*

In other words, any property which holds for the Baire space and is preserved under continuous maps holds for any Polish space. Second, the Baire space is useful in descriptive set theory, which studies the properties of the reals as a topological space, since:

Theorem 1.2 *The Baire space is homeomorphic to the set of irrational numbers.*

Proof Consider the map:

$$\begin{aligned} \varphi : \mathcal{N} &\to (0,1) \\ \langle x_i \rangle_{i\in\omega} &\mapsto \cfrac{1}{(x_0+1) + \cfrac{1}{(x_1+1)+\cdots}} \end{aligned}$$

which sends an infinite sequence of nonnegative integers to a continued fraction. As every irrational number in $(0, 1)$ has a unique expansion as a continued fraction of the above form, φ is a bijection between $\mathcal{N}$ and the irrationals in the interval $(0, 1)$. One can check that this is in fact a homeomorphism. Now since $(0, 1)$ is homeomorphic to $\mathbb{R}$, the claim follows. □

Hence $\mathbb{R}$ and $\mathcal{N}$ are ”homeomorphic up to a countable set”. We will see that the three properties we are going to study are invariant under removing or adding countable sets, and hence for practical purposes we will generally think of $\mathbb{R}, \mathcal{N} = \omega^\omega$ or ${}^\omega\omega$ as the real numbers, using whichever representation is most useful.

1.2 Borel sets

In the following, let S be a set. In this section we will construct the Borel hierarchy of a topological space, which can be thought of as the minimal set system extending the open sets where measure theory is well defined.

Definition 1.3 *A set system $\mathcal{A} \subset 2^S$ is called a:*

- *semi-ring if $\emptyset \in \mathcal{A}$, $\mathcal{A}$ is closed under finite intersections and for every $X, Y \in \mathcal{A}$ there always are $n \in \omega$ and $Z_i \in \mathcal{A}$ $(0 \leq i \leq n)$ such that $X \setminus Y = \sqcup_{i=0}^{n} Z_i$.*
- *ring if $\emptyset \in \mathcal{A}$ and $\mathcal{A}$ is closed under set differences and finite unions.*
- *σ-algebra if $X \in \mathcal{A}$ and $\mathcal{A}$ is closed under complements and countable unions.*

Remark *There are also the notions of σ-ring and algebra, however we will not consider them.*

Every σ-algebra is a ring and every ring is a semi-ring, hence these notions are increasing in complexity.

Example 1.1 *Let $S = \mathbb{R}$.*

- *$\mathcal{R}_0 = \{[a, b) : a, b \in \mathbb{R}\} \subset 2^{\mathbb{R}}$ is a a semi-ring, but not a ring*
- *$\mathcal{R} = \{\sqcup_{i=0}^{n} [a_i, b_i) : a_i, b_i \in \mathbb{R}\} \subset 2^{\mathbb{R}}$ is a a ring, but not a σ-algebra. It can be shown that it is in fact the smallest ring containing $\mathcal{R}_0$.*
- *The biggest σ-algebra which contains $\mathcal{R}$ is simply $2^{\mathbb{R}}$.*

The smallest σ-algebra which contains $\mathcal{R}$ is called the *Borel σ-algebra* on $\mathbb{R}$ and denoted by $\mathcal{B} = \mathcal{B}(\mathbb{R})$. Explicitely it is given by:

$$\mathcal{B} = \bigcap_{\substack{\mathcal{R} \subset \mathcal{A} \\ \mathcal{A} \text{ is a } \sigma\text{-algebra}}} \mathcal{A}$$

It exists, since the intersection of all σ-algebras containing $\mathcal{R}$ (which by the previous point is non-empty) is again a σ-algebra, and certainly minimal among all such objects.

How are the Borel sets structured? For this it is useful to take a more abstract approach and look at an equivalent definition of Borel sets, for general topological spaces S. Here we define $\mathcal{B}(S)$ to be the smallest σ-algebra which contains all the open sets. It turns out they can be arranged in a hierarchy of sets with increasing complexity.

Definition 1.4 (Hierachy of Borel sets) *Define for ever countable ordinal α:*

- $\Sigma_1^0 = \{O : O \subset S \text{ open}\}$
- $\Pi_1^0 = \{O^c : O \in \Sigma_1^0\}$
- $\Sigma_\alpha^0 = \{\bigcup_{n \in \omega} A_n : A_n \in \Pi_{\beta_n}^0 \text{ for some } \beta_n < \alpha\}$
- $\Pi_\alpha^0 = \{A^c : A \in \Sigma_\alpha^0\}$

Theorem 1.3 *We have that:*

$$\mathcal{B}(S) = \bigcup_{\alpha < \omega_1} \Sigma_\alpha^0 = \bigcup_{\alpha < \omega_1} \Pi_\alpha^0$$

Proof The second equality holds because $\Pi^0_\alpha \subset \Sigma^0_\beta$ for $\alpha < \beta$, and $\Sigma^0_\alpha \subset \Pi^0_\beta$ when $\alpha + 1 < \beta$. Furthermore $\bigcup_{\alpha<\omega_1} \Sigma^0_\alpha = \bigcup_{\alpha<\omega_1} \Pi^0_\alpha$ is closed under complements (by definition of Π^0_α) and under countable unions (by regularity of ω_1, every countable sequence of sets in the union is contained in some Π^0_β for $\beta < \omega_1$), hence a σ-algebra. Also, because it contains all the open sets Σ^0_1, it is a σ-algebra containing all open sets, hence by definition it must include all Borel sets. On the other hand, if the inclusion was strict, there would be a minimal $\alpha < \omega_1$ such that Σ^0_α contains a non-Borel set $A = \bigcup_{n\in\omega} A_n$, where $A_n \in \Pi^0_{\beta_n}$. Since the Borel sets are closed under countable unions, one of the $A_n \in \Pi^0_{\beta_n}$ has to be non-Borel. Now since the Borel sets are also closed under complements, there is $\beta_n < \alpha$ and $\tilde{A} = A_n^c \in \Sigma^0_{\beta_n}$ such that $\tilde{A}$ is non-Borel. However this contradicts the minimality of α, and hence equality holds.□

It is interesting to know, although not important for our purposes that the Borel hierarchy is a strict hierarchy for sufficiently complicated spaces.

Theorem 1.4 *If S is an uncountable space, then this hierarchy is non-degenerate, meaning that for every countable ordinal α:*

$$\Sigma^0_\alpha \setminus \Pi^0_\alpha \neq \emptyset$$

To sum up, for a topological space S, the Borel sets $\mathcal{B}(S)$ are the minimal extension of the open sets which is complete under countable unions and under complements. They can be arranged in a non-degenerate hierarchy, which also gives a way to construct any one of them iteratively in countably many steps. This we will use later in the form of Borel codes.

1.3 Lebesgue measurability

The Lebesgue measure is a natural way to define the size of subsets of the real line. In the previous section we saw an evolution in complexity of subsets of the real line as follows:

$$\mathcal{R}_0 \rightsquigarrow \mathcal{R} \rightsquigarrow \Sigma^0_\alpha, \Pi^0_\alpha \rightsquigarrow \mathcal{B} \rightsquigarrow \mathcal{P}(\mathbb{R})$$

Here the arrows are not strict inclusions, since for instance $\mathcal{R}_0 \subsetneq \Sigma^0_1$. However, it encapsulates how the Lebesgue measure on $\mathbb{R}$ will be constructed. We will start out by defining what measure elements of $\mathcal{R}_0$ should have.

Definition 1.5 *Define the Lebesgue pre-content to be the map:*

$$\begin{aligned} \lambda_0 : \mathcal{R}_0 &\to [0, \infty) \\ [a, b) &\mapsto |b - a| \end{aligned}$$

It sends intervals to their natural length. We will now extend it to the Lebesgue measure trough a number of intermediate steps.

Definition 1.6 *Let S be a set, $\mathcal{R} \subset \mathcal{P}(S)$ be a ring and $\mu : \mathcal{R} \to [0, \infty]$ be a map. Then μ is called*

- *a content if it is additive, i.e. for $A, B \in \mathcal{R}$ disjoint, we have*

$$\mu(A \sqcup B) = \mu(A) + \mu(B)$$

- *a pre-measure if it is a content and σ-additive, i.e. for $(A_n)_{n\in\omega} \subset \mathcal{R}$ disjoint with $\sqcup_{n\in\omega} A_n \in \mathcal{R}$ we have*

$$\mu\left(\bigsqcup_{n\in\omega} A_n\right) = \sum_{n\in\omega} \mu(A_n)$$

- *a measure, if it is a pre-measure and $\mathcal{R}$ is a σ-algebra.*

The following proposition now tells us that using only this data, we can extend the definition of λ_0 uniquely to the ring $\mathcal{R}$ generated by $\mathcal{R}_0$.

Proposition 1.1 *Let μ_0 be an additive map on a semi-ring $\mathcal{R}_0$. Then there is a unique additive map $\tilde{\mu} : \mathcal{R} \to [0, \infty)$ on the ring $\mathcal{R} = \{\bigcup_{i=1}^n A_i : A_i \in \mathcal{R}_0\}$ generated by $\mathcal{R}_0$, which extends μ_0.*

In particular, there is a unique additive map $\tilde{\lambda} : \mathcal{R} \to [0, \infty)$ extending λ. It is called the Lebesgue content.

Now we are able to assign a size to each finite collections of intervals, which is just the sum of their lengths. In fact, $\tilde{\lambda}$ is even a pre-measure, hence we can alternatively call it the *Lebesgue pre-measure*. It also has the property of *σ-finiteness*, which means we can write $\mathbb{R}$ as a countable collection of intervals, where each interval has finite Lebesgue pre-measure.

The next step is to extend the result to the Borel σ-algebra.

Theorem 1.5 (Carathéodory extension theorem) *Let $\tilde{\mu}$ be a σ-finite pre-measure on a ring $\mathcal{R} \subset 2^S$. Then there exists a unique measure μ on the smallest σ-field containing $\mathcal{R}$, extending $\tilde{\mu}$.*

Applying this result to the Lebesgue pre-measure, we get a unique map:

$$\lambda : \mathcal{B} \to [0, \infty]$$

This is the *Lebesgue measure* for Borel sets. We can extend one last time to the appropriately named *Lebesgue measurable sets*.

Definition 1.7 *Define the Lebesgue outer measure λ^* to be the map:*

$$\begin{aligned} \lambda^* : \mathcal{P}(S) &\to [0, \infty] \\ X &\mapsto \inf \left\{ \sum_{n=1}^{\infty} \lambda_0(A_n) : A_n \in \mathcal{R}_0 \text{ and } C \subset \bigcup_{n=1}^{\infty} \right\} \end{aligned}$$

We say that a set $A \subset \mathbb{R}$ *is Lebesgue measurable if for all* $C \in \mathcal{P}(\mathbb{R})$ *we have:*

$$\lambda^*(C) = \lambda^*(C \cap A) + \lambda^*(C \cap A^{\mathsf{c}})$$

We denote the set of all Lebesgue measurable sets by $\mathcal{L} = \mathcal{L}(\mathbb{R})$. *Furthermore, a set* $A \subset \mathbb{R}$ *is called null, if* $\lambda^*(A) = 0$.

In fact, the measurable sets differ not by a lot from the Borel sets.

Proposition 1.2 *A set* $A \subset \mathbb{R}$ *is Lebesgue measurable, if and only if there is a Borel set* $B \in \mathcal{B}$ *such that* $A \Delta B$ *is a null set.*

In particular this proposition implies that Lebesgue null sets are exactly the subsets of Borel null sets, the sets in $\mathcal{L} \setminus \mathcal{B}$ ”fill in the gaps” by giving every subset of a null set measure zero.

Proposition 1.3 *The set of all Lebesgue measurable sets* $\mathcal{L}$ *is a* σ*-algebra containing all intervals, and there is a unique extension of* λ *to* $\mathcal{L}$*, which by abuse of notation we will also call* λ.

The set function λ defined on $\mathcal{L}$ is what we will from now on refer to as the Lebesgue measure. Given the data that intervals should be sent to their natural length, we arrived without further input at it. It is a theorem of measure theory that λ is translation invariant. Hence it is the only measure on $\mathcal{L}$ which is translation invariant and sends intervals to their natural length. Furthermore, the Lebesgue measure can be approximated arbitrarily well using open and closed sets.

Proposition 1.4 ([3] on p.13) *For any Lebesgue measurable set* $A \subset \mathbb{R}$ *and* $\varepsilon > 0$ *there is a closed set* C *and an open set* O *such that* $C \subset A \subset O$ *and* $\lambda(O \setminus C) < \varepsilon$.

Finally, we note that at each step of extension, we added non-trivial sets to the domain of definition of the measure. However, if we assume the axiom of choice, there are still sets we are not able to measure.

Proposition 1.5 *We have the following inclusions:*

$$\mathcal{R}_0 \subsetneq \mathcal{R} \subsetneq \mathcal{B} \subsetneq \mathcal{L} \subset \mathcal{P}(\mathbb{R})$$

If we assume **AC***, we even have that* $\mathcal{L} \subsetneq \mathcal{P}(\mathbb{R})$.

Proof All the inclusions have already been proven or are trivial, except for $\mathcal{L} \subsetneq \mathcal{P}(\mathbb{R})$ if **AC** is assumed. Define the following equivalence relation on $\mathbb{R}$:

$$a \sim b :\Leftrightarrow a - b \in \mathbb{Q}$$

Since every equivalence class has a member in $[0, 1]$, we can construct with the axiom of choice a system of representatives $V = \{a_\lambda : \lambda < \mathfrak{c}\} \subset [0, 1]$ for $\sim$. The set V

is called a *Vitali set*. We show that this set is not measurable. Let $(q_k)_{k\in\omega}$ be an enumeration of the rationals in $[0,1]$. Define:

$$V_k = V + q_k$$

The sets $(V_k)_{k\in\omega} \subset [-1,2]$ are mutually disjoint by definition of V, furthermore $[0,1] \subset \bigcup_{k\in\omega} V_k$, since every $x \in [0,1]$ belongs to some equivalence class of $\sim$, and hence can be written as $x = v + r$ with $v \in V$ and $r \in \mathbb{Q} \cap [0,1]$. Hence by σ-additivity we have:

$$1 \leq \sum_{k\in\omega} \lambda(V_k) \leq 3$$

However, by translation invariance, we have $\lambda(V_k) = \lambda(V)$, so that

$$1 \leq \sum_{k\in\omega} \lambda(V) \leq 3$$

which is absurd. Hence V is not Lebesgue measurable. □

1.4 Property of Baire

Definition 1.8 *Let X be a Polish space and $A \subset X$ be a subset. We call A*

- *nowhere dense if its closure has empty interior.*
- *meagre if is the countable union of nowhere dense sets*

Furthermore, we say that A has the Baire property or is almost open, if there is an open set $G \subset X$ such that $A\Delta O$ is meagre.

The notion of meagreness tries to capture the idea of topological insignificance of a set in analogy to null sets in measure theory. Both are variants of the idea of smallness. With this interpretation in mind, the Baire property says that a subset of a Polish space behaves almost like an open set in a topological sense, hence the alternative name. However on $\mathbb{R}$, they encapsulate quite different ideas of smallness, as demonstrated by the *fattened rationals*.

Proposition 1.6 *Let $(q_k)_{k\in\omega}$ be an enumeration of the rationals. Then the fattened rationals are defined to be:*

$$\mathbb{Q}_f = \bigcap_{i\in\omega} Q_i \qquad \textit{where } Q_i = \bigcup_{k\in\omega} (q_k - 2^{-k-i}, q_k + 2^{-k-i})$$

Then $\mathbb{Q}_f$ is a null-set and $\mathbb{Q}_f^c$ is meagre. In particular, $\mathbb{R}$ is the disjoint union of a null set and a meagre set.

Proof We have that $\mathbb{Q}_f$ is a measurable set as intersection of measurable sets. Furthermore: $\lambda(\mathbb{Q}_f) \leq \inf_{i\in\omega} \lambda(Q_i) \leq \inf_{i\in\omega} 2^{-i+1} = 0$. Now Q_i are open and dense. So Q_i^c are closed an nowhere dense. It follows that $\mathbb{Q}_f^c = \bigcup_{i\in\omega} Q_i^c$ is meagre. □

Proposition 1.7 *The sets with the Baire property form a σ-algebra which contains every open set. Hence Every Borel set has the Baire property.*

Proof It is clear that open sets have the Baire property. If $A \subset X$ has the Baire property there exists an open set $O \subset X$ such that $A\Delta O$ is meagre. Now consider $A^{\mathsf{c}}\Delta\bar{O}^{\mathsf{c}} = A\Delta\bar{O}$. The last set is meagre, since $\bar{O} \setminus O$ is nowhere dense. So the sets with the Baire property are closed under complements.

If we have sets $(A_n)_{n\in\omega}$ with the Baire property, and $(O_n)_{n\in\omega}$ open sets such that the $A_n\Delta O_n$ are meagre, then since $(\bigcup_{n\in\omega} A_n)\Delta(\bigcup_{n\in\omega} O_n) \subset \bigcup_{n\in\omega} A_n\Delta O_n$, where the last set is meagre, and subsets of meagre sets are meagre, $\bigcup_{n\in\omega} A_n$ also has the Baire Property. Hence the sets with the Baire property form a σ-algebra, and hence all Borel sets have the Baire property. □

While the property of Baire and Measurability are quite different properties, their treatment will be quite similar, in part due to the following approximation property, which parallels proposition 1.4.

Proposition 1.8 *Let $A \subset {}^{\omega}\omega$ have the Baire property. Then there is a G_δ set X and an F_σ set y such that $X \subset A \subset Y$ and $Y \setminus X$ is meagre.*

Proof First, the closure of a nowhere dense set is nowhere dense, since
$interior\bar{\bar{N}} = \bar{N}^{\mathrm{o}} = \emptyset$ if N is a nowhere dense set. Second, suppose that $O \subset \mathbb{R}$ is open and $A\Delta O = \bigcup_{k\in\omega} N_k$ with N_k nowhere dense. Then

$$A\Delta O \subset \overline{\bigcup_{k\in\omega} N_k} = \bigcup_{k\in\omega} \bar{N}_k =: F$$

where F is a meagre F_δ set. Then $X = O \setminus F$ is a G_δ set with the following properties:

$X \subset A$, since $A \setminus O \subset F$
$A \setminus X = (A \setminus O) \cup (A \cap F) \subset (A\Delta O) \cup F$ is meagre.

Consider $B = {}^{\omega}\omega \setminus A$, which also has the Baire property. Hence we can construct a G_δ set $\tilde{Y}$ with the same properties, so that if we set $Y = \tilde{Y}^{\mathsf{c}}$ we conclude:

$$X \subset A \subset Y \quad \text{and} \quad Y \setminus X \text{ is meagre.}$$ □

Finally, the last analogy to Lebesgue measurability is the fact that there are sets not possessing the property of Baire.

Proposition 1.9 **AC** *implies that there is a set of reals without the Baire property.*

Proof Consider the Vitali set V from proposition 1.5. Suppose it had the Baire property. Then there is $O \subset [0,1]$ open such that $O \Delta V = M$ is meagre. In particular, as O is a countable union of intervals, there is an interval $(a,b) \subset [0,1]$ such that $(a,b) \cap V^{c} = (a,b) \setminus V$ is meagre, since V_n and V are disjoint if $q_n \neq 0$, we have in particular that $(a,b) \cap V_n \subset (a,b) \cap V^{c}$ is meagre for $q_n \neq 0$. This means that $(a - q_n, b - q_n) \cap V$ is meagre for all $q_n \neq 0$, since meagreness is translation invariant. So in particular V is meagre, and every V_n as well. However, since

$$[0,1] \subset \bigcup_{n \in \omega} V_n,$$

so that the union of countably many meagre sets is not meagre, we have a contradiction. So V cannot have the Baire Property. □

1.5 Perfect set property

In this section we investigate the last regularity property of subsets of the real line, the perfect set property.

Definition 1.9 *A closed subset C of a Polish space X is called perfect if it admits no isolated points.*

Perfect subsets where first investigated in relation to the continuum hypothesis, due to the following property:

Proposition 1.10 *Every perfect subset of $\mathbb{R}$ has the cardinality of the continuum.*

By this proposition, a counter-example to the continuum hypothesis cannot be a perfect set. Next, let us explain where the name "perfect set" comes from.

Definition 1.10 *Let C be a closed subset of $\mathcal{N}$. We define the tree corresponding to C to be:*

$$T_C := \{s \in \omega^{<\omega} : \exists f \in C(s \preceq f)\}$$

So T_C contains all the finite subsequences of elements of C. We say that T_C is perfect, if it splits indefinitely, i.e. for every $s \in T_C$, there are $r, t \in T_C$ such that $s \preceq r$, $s \preceq t$, but $r \npreceq t$ and $t \npreceq r$, so r and t are incompatible.

Proposition 1.11 *A closed subset $C \subset {}^{\omega}\omega$ is perfect exactly when the corresponding tree T_C is a perfect tree.*

Proof An isolated point $f \in C$ is an element $f \in {}^{\omega}\omega$ such that there is an open neighborhood $O(s)$ with $s \in {}^{<\omega}\omega$ and $\{f\} = O(s) \cap C$. Since $O(s)$ corresponds to the subtree of T_C in which every element has the common prefix s, this means that starting at the node $s \in T_C$, the tree does not branch any further. Hence it is not perfect.

On the other hand, if the tree is perfect, then in any given subtree $T_{O(s)}$ there have to be at least two distinct branches, so that no point is isolated. □

In other words perfect subsets of $\mathbb{R}$ correspond to perfect trees, which explains their name.

The investigation to the continuum hypothesis through perfect sets can by extended by the perfect set property:

Definition 1.11 *A subset A of a Polish space X has the perfect set property if it is either countable or it contains a perfect set.*

The Cantor–Bendixson theorem states that every closed subset of a polish space can be written uniquely as a perfect set and a countable set, so that closed sets in polish spaces have the perfect set property in a strong sense. Hence a counterexample to the continuum hypothesis must necessarily be non-closed. However, this approach of showing that more and more complicated sets cannot be counterexamples by the perfect set property must ultimately fail in the presence of **AC**, as demonstrated by the following proposition.

Proposition 1.12 **AC** *implies that there is a set of reals without the perfect set property.*

Proof The Bernstein sets are certain subsets of the real line that intersect any uncountable closed subset of the reals but do not contain any such set in its entirety. A theorem of Bernstein states that such sets exist if **AC** is assumed. Since they have to be uncountable and don't contain any uncountable closed subsets, they cannot entirely contain any perfect set, which are of cardinality $\mathfrak{c}$ by proposition 1.10. Hence they are examples of sets which do not have the perfect set property. □

Chapter 2

The Solovay Model

In this chapter, we construct a model of set theory, in which every subset of the real line is Lebesgue measurable, hence where the measure problem for $\mathbb{R}$ has a solution. To do this however, we need to abandon the axiom of choice **AC** and content ourselves with the weaker *axiom of dependent choice* **DC**. Instead of allowing for arbitrary families of sets, it restricts to the case of countable families. However, it is stronger than the *axiom of countable choice* $\mathbf{AC}_\omega$, as the choices need not be independent from one another. Formally it can be stated as follows:

$$\begin{aligned}\mathbf{DC}: \forall X \forall R((X \neq \emptyset \wedge R \subset X \times X \wedge \forall x \in X \exists y \in X(\langle x, y\rangle \in R)) \\ \rightarrow \exists f \in {}^{\omega}X \forall n \in \omega(\langle f(n), f(n+1)\rangle \in R))\end{aligned}$$

Furthermore, in this model all sets of reals have the Baire property and the perfect set property. More precisely, we are going to prove the main result of the text:

Theorem 2.1 (Solovay) *Suppose that κ is an inaccessible cardinal and G is $\mathrm{Col}(\omega,\kappa)$-generic. Then $\mathbf{V}[G]$ has an inner model satisfying:*

- *Every set of reals is Lebesgue measurable.*
- *Every set of reals has the Baire property.*
- *Every set of reals has the perfect set property.*
- *The Principle of Dependent Choice* **DC**.

Before proceeding to the proof however, we need to recall a few notions from logic and set theory, most prominently the concept of forcing.

2.1 Preliminary notions

Inner Models

An *inner model* M of the set-theoretic universe $\mathbf{V}$ a proper transitive class that contains all the ordinals and satisfies the axioms of **ZF**. The archetypical example of an inner model is the *constructible universe* L. We say that a set y is *definable* over a set x if there is a first order formula $\varphi(v_0, \ldots, v_n)$ and parameters $a_1, \ldots, a_n \in x$ such that:

$$z \in y \Leftrightarrow \langle x, \in \rangle \vDash \varphi(z, a_1, \ldots, a_n)$$

where $\langle x, \in \rangle$ denotes the set model with domain x. Define for any set x:

$$\operatorname{def}(x) := \{y \subset x : y \text{ is definable over } x\} \subset \mathcal{P}(x)$$

This is itself a set, since the satisfaction relation for $\langle x, \in \rangle$ can be formalized inside the universe. The constructible universe is now given by transfinite induction:

$$L_0 = \emptyset; \quad L_{\alpha+1} = \operatorname{def}(L_\alpha); \quad L_\gamma = \bigcup_{\alpha<\gamma} L_\alpha$$

and $L := \bigcup_{\alpha\in\mathbf{On}} L_\alpha$. Intuitively, it is the smallest inner model of $\mathbf{V}$ since every set that can be described explicitly must be part of any inner model. The construction mirrors to the construction of the von Neumann universe $\mathbf{V} = \bigcup_{\alpha\in\mathbf{On}} \mathbf{V}_\alpha$, but instead of the full powerset $\mathcal{P}(\mathbf{V}_\alpha)$, only the "unavoidable" *constructible* sets $\operatorname{def}(L_\alpha)$ are considered. Another property of L that will be useful later is that it can be well-ordered, level by level. The first level L_0 is trivially well-ordered, and by transfinite recursion, the set $L_{\alpha+1} \setminus L_\alpha$ can be well-ordered by the lexicographical order of formulae and the well-ordering on the parameters.

Two further examples of inner models are given for any set A by the *constructible closure* $L(A)$ and the inner model $L[A]$ of *sets constructible relative to* A.

The constructible closure is the smallest inner model which contains A. Explicitly it is given by:

$$L_0(A) = \operatorname{TC}(\{A\}); \quad L_{\alpha+1}(A) = \operatorname{def}(L_\alpha(A)); \quad L_\gamma(A) = \bigcup_{\alpha<\gamma} L_\alpha(A)$$

and $L(A) := \bigcup_{\alpha\in\mathbf{On}} L_\alpha(A)$. It can be shown that $L(A)$ satisfies **AC** exactly when $\operatorname{TC}(\{A\})$ has a well-ordering which is present in $L(A)$. Furthermore, we can construct a surjective class function:

$$\Phi : L \times A \to L(A)$$

such that $\Phi|_{\alpha\times A} \in L(A)$, by first constructing a surjection onto $\operatorname{TC}(\{A\})$, and after that proceeding as for the well-ordering of L, going level by level via transfinite induction.

On the other hand, there is the model $L[A]$ which consists of all the sets whose definition can contain assertions about membership in A. The modified definition relation is given by:

$$\operatorname{def}^A(x) := \{y \subset x : y \text{ is definable over } x \text{ using the unary relation } R_{A\cap x}\} \subset \mathcal{P}(x)$$

Here $R_{A\cap x}(y)$ hold exactly when $y \in A \cap x$. We then define:

$$L_0[A] = \emptyset; \quad L_{\alpha+1}[A] = \mathrm{def}^A(L_\alpha[A]); \quad L_\gamma[A] = \bigcup_{\alpha<\gamma} L_\alpha[A]$$

and $L[A] = \bigcup_{\alpha\in\mathbf{On}} L_\alpha[A]$. Exactly like L, we can well-order $L[A]$ using transfinite induction.

To get a feeling for these models, consider the set A. It is part of $L(A)$ by definition, however in $L[A]$ we may only have a part of A. For instance $L[\mathbb{R}] = L \subsetneq L(\mathbb{R})$.

Inaccessible cardinals

An *inaccessible* cardinal is a cardinal κ such that:

- κ is an uncountable limit cardinal,
- κ is regular,
- for any $\lambda < \kappa$ we have $2^\lambda < \kappa$.

Inaccessible cardinals κ cannot be attained from below neither by taking unions of fewer than κ sets (by regularity) nor by considering power sets (which is exactly the third property). Hence the name inaccessible. Interestingly, their existence cannot be proven in **ZFC**, since:

Proposition 2.1 *Let κ be inaccessible and $\mathbf{V}_\kappa$ be the κ-stage in the von Neumann hierarchy. Then:*

$$\mathbf{V}_\kappa \vDash \mathbf{ZFC}$$

If the existence of such κ could be proven, the existence of a minimal κ could be proven as well. Then $\mathbf{V}_\kappa$ is a model of **ZFC**, hence it is also a model for the existence of an inaccessible cardinal λ. One easily checks that inaccessibility in $\mathbf{V}_\kappa$ and $\mathbf{V}$ are the same concept (since they have the same elements and power sets), so that λ is also inaccessible in $\mathbf{V}$. However $\kappa \notin \mathbf{V}_\kappa$, so that $\lambda < \kappa$, contradicting the minimality of κ.

The sheer size of such inaccessible cardinals is essential to the proof of the main theorem 2.1, as their inaccessibility allows to overshadow unwanted behaviour in the ground model.

Lévy Hierarchy of Formulas

If φ is a formula in the language of set theory, and M is an inner model, then the formula φ^M denotes the *relativisation* of φ to M, i.e. the formula obtained by restricting the quantifiers to M. We say that a formula $\varphi(x_1, \ldots, x_n)$ is *absolute for M*, if for any choice of $x_1, \ldots, x_n \in M$:

$$\mathbf{V} \vDash \varphi(x_1, \ldots, x_n) \Leftrightarrow M \vDash \varphi^M(x_1, \ldots, x_n).$$

Hence the truth value of φ can be determined inside the smaller model M, without needing to consider the rest of $\mathbf{V}$.

We say that a formula φ is Σ_0 and Π_0 if it does not contain unbounded quantifiers, i.e. every quantifier that appears in φ is restricted to a set. For $n \in \omega$, a formula is Σ_{n+1} if it is equivalent to $\exists v_0 \exists v_1 \ldots \exists v_k \varphi$ with φ being Π_n, and it is Π_{n+1} if it is equivalent to $\forall v_0 \forall v_1 \ldots \forall v_k \varphi$ with φ being Σ_n. A formula is Δ_n if it is both Σ_n and Π_n. It is clear by the existence of the Prenex normal form that every formula belongs to some level of this so called *Lévy Hierarchy*. In particular, we are interested in Δ_0 and Δ_1 formulas, since:

Proposition 2.2 *Let $\varphi \in \Delta_0 \cup \Delta_1$ and M be an inner model. Then φ is absolute for M.*

For Δ_0 formulas it holds, since the bounded quantifiers all run over sets already present in M. For Δ_1 formulas, the following intuition is given as a moral proof of why it should be true. By the Π_1-formulation of $\varphi \cong \forall v_0 \forall v_1 \ldots \forall v_k \varphi_0$, with φ_0 a Δ_0 formula, if φ holds in $\mathbf{V}$, we have that for every choice of the variables $v_0, \ldots, v_n$ in $\mathbf{V}$, the formula φ_0 holds. By absoluteness of Δ_0 formulas φ_0 also holds in M for all such choices, so φ holds in M as well. This is called *downward persistence*. By the Σ_1-formulation of $\varphi \cong \exists v_0 \exists v_1 \ldots \exists v_k \varphi_0$, with φ_0 a Δ_0 formula, if the formula φ^M holds in M, we have that for some choice of the variables $v_0, \ldots, v_n$ in M, the formula φ_0 holds. By absoluteness of Δ_0 formulas φ_0 also holds in $\mathbf{V}$ for that choices, so φ holds in $\mathbf{V}$ as well. This is called *upward persistence*.

The next proposition is used to verify **DC** in the inner model mentioned in the main theorem 2.1, which relies on the notion of well-foundedness being absolute.

Proposition 2.3 *Let R be a relation on a set X. Then the formula:*

$$\ulcorner R \textit{ is well-founded} \urcorner$$

is Δ_1 and hence absolute for inner models M.

The Notion of Forcing

In this text, a *forcing partial order* (forcing p.o.) is a partially ordered set $\mathbb{P} = (P, \leq)$ with a minimal element, denoted $\mathbf{0}$. For *conditions* $p, q \in P$, we say that:

$$q \leq p, \text{ if } p \text{ extends } q$$

so that p is more restrictive (informative) than q.

Let G be a $\mathbb{P}$-generic filter. We denote unspecific $\mathbb{P}$-names for elements $x \in \mathbf{V}[G]$ by $\underset{\sim}{x}$, so that the interpretation of the name in the extension yields $\underset{\sim}{x}[G] = x$. For elements in the ground model $x \in \mathbf{V}$, we denote its canonical name by $\underset{\cdot}{x}$, so that $\underset{\cdot}{x}[G] = x \in \mathbf{V}[G]$. Notice the difference between:

$$\underset{\cdot}{\omega}_{27} \text{ and } \underset{\sim}{\omega}_{27},$$

the former being the canonical name of the 27-th uncountable cardinal in the ground model, and the latter being a name for the 27-th uncountable cardinal in the extension. In other words, $\check{\omega}_{27}[G]$ is the image of $\omega_{27} \in \mathbf{V}$ under the inclusion $\mathbf{V} \to \mathbf{V}[G]$, while $\underset{\sim}{\omega}_{27}[G] \in \mathbf{V}[G]$ is the 27-th uncountable cardinal in $\mathbf{V}[G]$. We will furthermore denote the canonical name for the generic filter G by $\dot{G}$. The forcing relation ship will be denoted by $p \Vdash_{\mathbb{P}} \varphi$ (or $p \Vdash \varphi$ if the p.o. is clear from context), where p is a condition, and φ a formula in the *forcing language*, i.e. a formula with $\mathbb{P}$-names as parameters. We will use the shorthand $\Vdash \varphi$ for $\mathbf{0} \Vdash \varphi$.

A partial order $\mathbb{P}$ is called *separative*, if for every $p, q \in P$ with $p \ngeq q$ there is an $r \in P$ with $r \geq p$ and $r \perp q$. This means that the partial order never stops branching out, and every two conditions, even if they are not directly related by an order relation can be distinguished when looking further up the partial order. Although not every forcing p.o. is separative, similarly to antisymmetry every p.o. can be made separative up to a dense embedding by considering its *separative quotient*:

Proposition 2.4 *For every forcing partial order $\mathbb{P}$, we can define the equivalence relation : $p \sim q \Leftrightarrow \forall r \in P(r \parallel p \leftrightarrow r \parallel q)$. The set of equivalence classes :*

$$\tilde{P} = P/\sim$$

together with the order relation:

$$[p] \geq [q] \Leftrightarrow \forall p \in P(r \geq p \to r \parallel q)$$

is called the separative quotient of $\mathbb{P}$ and is separative. The obvious projection map is a dense embedding.

We rely on a few results about forcing, the first one being about the preservation of inaccessibility in forcing extensions when the forcing p.o. is small enough.

Proposition 2.5 *Let κ be an inaccessible cardinal and $\mathbb{P}$ a forcing p.o. such that $|\mathbb{P}| < \kappa$. Then*

$$\Vdash_{\mathbb{P}} \kappa \text{ is inaccessible}$$

i.e. κ is still inaccessible in any forcing extension when forcing with $\mathbb{P}$.

Next, we remind the reader of the Δ-system lemma, which is later used to prove chain conditions of forcing p.o.'s.

Theorem 2.2 (Δ-System Lemma, theorem 1.6 in [4, p.49]) *Let λ be any infinite cardinal. Let $\kappa > \lambda$ be regular and satisfy $\forall \alpha < \kappa(\alpha^{<\lambda} < \kappa)$. Assume $|A| \geq \kappa$ and for each $x \in A$ we have $|x| < \lambda$. Then there is a Δ-system $B \subset A$ of cardinality κ. In other words, there is an $r \in B$ such that $\forall x, y \in B : (x \cap y = r)$*

We can specialize to the case where κ is inaccessible, as $\forall \alpha < \kappa$ we have $\alpha^{<\lambda} < 2^{\alpha^{<\lambda}} \leq 2^{\alpha\lambda} < \kappa$. The last inequality uses inaccessibility and the fact that $\alpha\lambda < \kappa$.

Corollary 2.1 *Let λ be any infinite cardinal. Let $\kappa > \lambda$ be inaccessible. Assume $|A| \geq \kappa$ and for each $x \in A$ we have $|x| < \lambda$. Then there is a Δ-system $B \subset A$ of cardinality κ. In other words, there is an $r \in B$ such that $\forall x, y \in B : (x \cap y = r)$*

Finally a result about factoring a forcing extension through an inner model.

Proposition 2.6 *Suppose that $\mathbb{P}$ is a forcing p.o., G is $\mathbb{P}$-generic over $\mathbf{V}$, and $\mathbf{N}$ is a transitive model of* **ZFC** *such that $\mathbf{V} \subset \mathbf{N} \subset \mathbf{V}[G]$. Then there is a p.o. $\mathbb{Q}$ and a $\mathbb{Q}$-name $\underset{\sim}{\mathbb{S}}$ such that:*

$$\Vdash_{\mathbb{Q}} \ulcorner \underset{\sim}{\mathbb{S}} \text{ is a forcing p.o. with } \mathbb{Q} * \underset{\sim}{\mathbb{S}} \text{ isomorphic to } \mathbb{P} \urcorner$$

Identify now $\mathbb{P}$ with $\mathbb{Q} * \underset{\sim}{\mathbb{S}}$ and set:

$$G_0 = \{q \in Q : \exists \underset{\sim}{s}(\langle q, \underset{\sim}{s}\rangle \in G)\}$$
$$G_1 = \{\underset{\sim}{s}[G_0] : \exists q(\langle q, \underset{\sim}{s}\rangle \in G)\}$$

Then G_0 is $\mathbb{Q}$-generic with $\mathbf{V}[G_0] = \mathbf{N}$ and G_1 is $\underset{\sim}{\mathbb{S}}[G_0]$-generic over $\mathbf{N}$ with $\mathbf{N}[G_1] = \mathbf{V}[G]$.

2.2 Borel Codes

The goal of this section is to encode Borel sets via special real numbers, called *Borel codes*, so that we are able to reference Borel sets from the extension in the ground model and vice versa. This coding is possible since all Borel sets can be constructed from open sets in countably many steps by theorem 1.3 on the Borel Hierarchy. Pick a bijection $\Gamma : \omega^2 \to \omega$. And choose the countable basis $(I_n)_{n\in\omega} = (O(s))_{s\in{}^{<\omega}\omega}$ of the topology of the reals ${}^{\omega}\omega$. Consider a real $c \in {}^{\omega}\omega$ and define the following auxiliary maps:

$$d(n) = c(n+1)$$
$$d_i(n) = c(\Gamma(i, n) + 1)$$

The first entry in the code c is a flag number, signalling the type of code, and the rest is either a single code, denoted by d, or a countable family of codes, denoted by d_i for $i \in \omega$.

Definition 2.1 *Let $\alpha < \omega_1$. We call $c \in {}^{\omega}\omega$:*

- *A Σ_1-code if $c(0) > 1$.*
- *A Π_α-code if either c is already a Σ_β- or Π_β-code for some $\beta < \alpha$, or $c(0) = 0$ and d is a Σ_α-code*
- *A Σ_α-code with $\alpha > 1$ if either c is already a Σ_β- or Π_β-code for some $\beta < \alpha$, or $c(0) = 1$ and d_i for $i \in \omega$ are all Π_α-codes.*

We denote the collection of all Borel codes $\mathbf{BC} \subset {}^{\omega}\omega$.

We now define the Borel set that is coded by $c \in \mathbf{BC}$, in complete analogy to the construction of the hierarchy.

$$A_c = \begin{cases} \bigcup\{I_n : d(n) = 1\} & \text{, if } c \text{ is a } \Sigma_1\text{-code} \\ (A_d)^c & \text{, if } c \text{ is a } \Pi_\alpha\text{-code} \\ \bigcup_{i\in\omega} A_{d_i} & \text{, if } c \text{ is a } \Sigma_\alpha\text{-code with } \alpha > 1 \end{cases}$$

We call a Σ_1-code an *open* code, a Π_1-code a *closed* code, a Σ_2-code an F_σ code, a Π_2-code an G_δ code, a Σ_3-code an $G_{\delta\sigma}$ code and a Π_3-code an $F_{\sigma\delta}$ code, by referring to the type of set they encode.

It is clear from theorem 1.3 on the Borel Hierarchy that the sets A_c generated by these codes $c \in {}^{\omega}\omega$ exhaust all Borel sets:

Proposition 2.7 $\{A_c : c \in \mathbf{BC}\} = \mathcal{B}$

Furthermore, Borel codes have a few properties that makes it easier to talk about them independently of the forcing extension.

Proposition 2.8 *The properties:*

$$\begin{array}{ll} x \in A_c; & A_c = \emptyset \\ A_l = A_c \cup A_d; & A_l = A_c \cap A_d \\ A_l = {}^{\omega}\omega \setminus A_c; & A_l = A_c \Delta A_d \\ A_l = \bigcup_{i\in\omega} A_{c_i} & \end{array}$$

are absolute for all transitive models of **ZF**.

Proof The sets I_n for $n \in \omega$ are satisfy all these properties, since the $O(s)$ depend only on the sequences s, hence all such statements about the I_n can be reduced to statements about finite sequences. Now, if the above statements are absolute for all Σ_α-codes, then they are also absolute for all $\Sigma_{\alpha+1}$-codes, since they can be reduced to statements about Σ_α-codes using the expansion of the codes. Hence the statements are absolute for all codes. □

Finally, through Borel codes we can prove that the Lebesgue measure is absolute for Borel sets.

Proposition 2.9 *Let M be an inner model and c a Borel code. Then*

$$\lambda^M(A_c^M) = \lambda(A_c)$$

Proof By the previous proposition, unions an disjoint unions are absolute for the inner model M. Since the value of λ and λ^M is entirely determined by its value on basic opens, we only need to prove that $\lambda^M(O^M(s)) = \lambda(O(s))$. However, the value of λ on basic opens depends only on the sequence $s \in {}^{<\omega}\omega$. □

2.3 The Levy collapse

Lemma 2.1 *Let λ be a regular cardinal and let $\alpha \geq \lambda$ be any cardinal. Then there is forcing notion $\mathbb{P}^{\lambda}_{\alpha} = \mathbb{P}_{\alpha}$, which adjoins a surjective map $f : \lambda \to \alpha$ not present in the base model, i.e. if G is $\mathbb{P}_{\alpha}$-generic, then:*

$$V[G] \vDash f : \lambda \twoheadrightarrow \alpha$$

Furthermore $\mathbb{P}_{\alpha}$ is λ-closed.

Proof Define the following set of partial functions:

$$P_{\alpha} = \{p : p \text{ is a function } \wedge \operatorname{dom}(p) \subset \lambda \wedge |\operatorname{dom}(p)| < \lambda \wedge \operatorname{ran}(p) \subset \alpha\}.$$

The desired partial order is then given by $\mathbb{P}_{\alpha} = (P_{\alpha}, \subset)$. Suppose G is $\mathbb{P}_{\alpha}$-generic. Then $f = \bigcup G$ is a well defined partial function $f \in \operatorname{Fn}(\lambda, \alpha)$, since G is a filter (and hence every two conditions are compatible). It is a total function, since for any $\mu \in \lambda$: $D_{\mu} = \{p \in P_{\alpha} : \mu \in \operatorname{dom}(p)\}$ is open dense. Furthermore, we have surjectivity, since for any $\beta \in \alpha$, $E_{\beta} = \{p \in P_{\alpha} : \exists \mu \in \lambda : (\mu \in \operatorname{dom}(p) \wedge p(\mu) = \beta)\}$ is open dense. Finally, we have to check that the adjoined function is not already present in the base model. Let $g : \lambda \to \alpha$ be any function in $\mathbf{V}$. Then define: $F_{g} = \{p \in P_{\alpha} : \exists \mu \in \lambda : (\mu \in \operatorname{dom}(p) \wedge p(\mu) \neq g(\mu))\}$, which is of course an open dense subset, and forces $V[G] \vDash f \neq g$.

Finally, we have λ-closedness, since a chain of length $\mu < \lambda$ is bounded by the union of its elements, and this bound is in $\mathbb{P}_{\alpha}$, since λ is regular. □

The forcing notion $\mathbb{P}_{\alpha}$ introduced above adjoins a surjective function $f : \lambda \twoheadrightarrow \alpha$. In the particular case of $\lambda = \alpha = \omega$ we have that $\mathbb{P}^{\omega}_{\omega}$ adjoins a surjective function $f : \omega \twoheadrightarrow \omega$, in other words a *Cohen real*. In this case we will denote it by $\mathbb{C}$. The following proposition demonstrates that it is in a sense the minimal forcing p.o. doing so in the case where $\lambda = \omega$.

Proposition 2.10 ([3, p.129]) *Let $\mathbb{P}$ be a separative p.o. such that $|P| < \alpha$, and:*

$$\Vdash_{\mathbb{P}} \exists f (f : \omega \twoheadrightarrow \alpha \textit{ is surjective } \wedge f \notin \check{\mathbf{V}})$$

Then there is an injective, dense embedding of a dense subset of $\mathbb{P}_{\alpha}$ into $\mathbb{P}$.

Notice that in the proof of the above proposition separativity is required.

For $\lambda < \alpha$, the forcing p.o. collapses α to λ. The next theorem uses this property and product forcing to adjust the size of a range of cardinals, leading to the central forcing notion of this thesis, the *Lévy Collapse*.

Theorem 2.3 (The Lévy Collapse) *Let λ be a regular cardinal and let $\kappa > \lambda$ be an inaccessible cardinal. Then there is a forcing notion $\operatorname{Col}(\lambda, \kappa)$, called the Lévy Collapse such that if G is $\operatorname{Col}(\lambda, \kappa)$-generic then:*

- *Every α such that $\lambda \leq \alpha < \kappa$ has cardinality λ in $\mathbf{V}[G]$.*
- *Every cardinal α with $\alpha \geq \kappa$ or $\alpha \leq \lambda$ is preserved.*

Hence $\mathbf{V}[G] \models \kappa = \lambda^+$.

Proof Define

$$\mathrm{Col}(\lambda, \kappa) = \prod_{\alpha<\kappa} \mathbb{P}_\alpha$$

□

as the λ-supported product of the forcing notions $\mathbb{P}_\alpha$, so that elements of $\mathrm{Col}(\lambda, \kappa)$ are supported on fewer than λ coordinates. The idea here is to adjoin a surjective function $f_\alpha : \lambda \twoheadrightarrow \alpha$ for every $\alpha < \kappa$.

In fact, if G is a $\mathrm{Col}(\lambda, \kappa)$-generic, then the projection G_α of G onto P_α is a $\mathbb{P}_\alpha$-generic filter. Define $f_\alpha = \bigcup G_\alpha$, so that $f_\alpha : \lambda \twoheadrightarrow \alpha$ is surjective and $\mathbf{V}[G_\alpha] \models |\alpha| \leq |\lambda|$, by the previous lemma. Hence also $\mathbf{V}[G] \models |\alpha| \leq |\lambda|$.

Consider a family of conditions: $F = \{p_\eta : \eta \in \kappa\} \subset \mathrm{Col}(\lambda, \kappa)$. We will employ the Δ-system lemma 2.1 to show that F cannot be an anti-chain. Hence $\mathrm{Col}(\lambda, \kappa)$ satisfies the κ-chain condition, so that all cardinals $\gamma \geq \kappa$ are preserved. Define for $p \in \mathrm{Col}(\lambda, \kappa)$ with projections $p_\alpha \in P_\alpha$ the domain of p to be: $\mathrm{dom}(p) = \bigsqcup_{\alpha\in\kappa} \mathrm{dom}(p_\alpha)$. Then for each $p \in F$, $|\mathrm{dom}(p)| < \lambda$, as λ is regular. Applying 2.1 to $D = \{\mathrm{dom}(p_\eta) : \eta \in \kappa\}$ we obtain a Δ-system $\{\mathrm{dom}(p) : p \in \tilde{F} \subset F\} = \tilde{D} \subset D$ of cardinality κ. So that $\exists r \in \bigsqcup_{\alpha\in\kappa} \lambda$ such that $\mathrm{dom}(p) \cap \mathrm{dom}(q) = r$ for all $p, q \in \tilde{F}$. Since $|r| < \kappa$, so that $|r|^\lambda < \kappa$, by the pigeonhole principle we can find $p, q \in \tilde{F}$ such that $p|_r = q|_r$. But then $p \parallel q$, and F cannot be an anti-chain.

Finally, $\mathrm{Col}(\lambda, \kappa)$ is λ-closed, since all the P_α are. Hence, all cardinals $\gamma \leq \lambda$ are preserved.

The next lemma states that ajoining countable sequences of ordinals does not require the full power of $\mathrm{Col}(\omega, \kappa)$, as the sub-partial order $\mathrm{Col}(\omega, \delta)$ for $\delta < \kappa$ suffices.

Lemma 2.2 *Let κ be inaccessible. For any $a \in \mathbf{V}[G]$ with $a : \omega \to \mathbf{On}$, there is a $\delta < \kappa$ such that $a \in \mathbf{V}[\pi_{<\delta}(G)]$. Here $\pi_{<\delta}(G)$ denotes the projection of G onto the coordinates α with $\alpha < \delta$.*

Proof Let $\underset{\sim}{a}$ be a name for $a \in \mathbf{V}[G]$. Consider for each $n \in \omega$ a maximal anti-chain $A_n \subset \mathrm{Col}(\omega, \kappa)$, such that for each $p \in A_n$, there is $\alpha \in \mathbf{On}$ with $p \Vdash \underset{\sim}{a}(n) = \alpha$. In other words, A_n should contain sufficiently many conditions, so that all possible values of $a(n)$ in an extension $\mathbf{V}[G]$ can be forced by them. Since $\mathrm{Col}(\omega, \kappa)$ has the κ-c.c., we get $\forall n \in \omega(|A_n| < \kappa)$ and hence by regularity of κ:

$$\Big|\bigcup_{n\in\omega} A_n\Big| < \kappa$$

Now, again by regularity of κ, there has to be a $\delta < \kappa$, such that if $p \in \bigcup_{n\in\omega} A_n$, then for any $\alpha < \kappa$ we have $\mathrm{ran}(p_\alpha) \subset \delta$. But then already $\pi_{<\delta}(p) \in \mathrm{Col}(\omega, \delta)$. Hence a

is definable from $\pi_{<\delta}(G)$ by setting

$$a(n) = \alpha \Leftrightarrow p \Vdash \underset{\sim}{a}(n) = \alpha$$

for the unique $p \in G \cap A_n$, hence $a \in \mathbf{V}[\pi_{<\delta}(G)]$. □

The next lemma states that countable sequences of ordinals which are present in a $\mathrm{Col}(\omega, \kappa)$-extension can be seen as an intermediate step. Alternatively, we could construct the extension by first adjoining the sequence and then finding another extension starting from the slightly bigger model.

Lemma 2.3 (Factoring lemma) *Let κ be inaccessible and G be* $\mathrm{Col}(\omega, \kappa)$*-generic. Then for any $a \in \mathbf{V}[G]$ with $a : \omega \to$* **On***, there is an H, which is* $\mathrm{Col}(\omega, \kappa)$*-generic over $\mathbf{V}[a]$ such that $\mathbf{V}[G] = \mathbf{V}[a][H]$.*

Proof (Sketch) By the previous lemma, there is $\delta < \kappa$ such that $a \in \mathbf{V}[G \cap \mathrm{Col}(\omega, \delta)]$. Define:

$$\begin{aligned} G_- &= \pi_{<\delta}(G) & \mathrm{Col}(\omega, <\delta) &= \mathrm{Col}(\omega, \delta) \\ G_0 &= \pi_\delta(G) & \mathrm{Col}(\omega, \{\delta\}) &= \mathbb{P}_\delta \\ G_+ &= \pi_{>\delta}(G) & \mathrm{Col}(\omega, >\delta) &= \prod_{\delta<\alpha<\kappa} \mathbb{P}_\alpha, \end{aligned}$$

where $\pi_{>\delta}(G)$ is defined to be the projection of G to the coordinates strictly bigger than δ, in complete analogy to $\pi_{<\delta}(G)$ from before. It is easy to see that:

$$\mathrm{Col}(\omega, \kappa) = \mathrm{Col}(\omega, <\delta) \times \mathrm{Col}(\omega, \{\delta\}) \times \mathrm{Col}(\omega, >\delta)$$

and that the $G_{-/0/+}$ are generic filters with respect to their corresponding p.o.'s.

By the factoring property of forcing extensions 2.6, there is a forcing notion $\mathbb{P} \in \mathbf{V}[x]$ and a $\mathbb{P}$-generic H_0 such that

$$\mathbf{V}[x][H_0] = \mathbf{V}[G_0]$$

Set $\mathbb{Q} = \mathbb{P} \times \mathrm{Col}(\omega, \{\delta\})$. Then $\mathbf{V}[x][H_0][G_1]$ is a forcing extension over $\mathbf{V}[x]$ using $\mathbb{Q}$, by the properties of product forcing. As $|Q| \leq \delta$, and by virtue of the second component of $\mathbb{Q}$, we have that the conditions of proposition 2.10 are fullfilled, so that there is a $\mathrm{Col}(\omega, \{\delta\})$-generic H_1 over $\mathbf{V}[x]$ such that:

$$\mathbf{V}[x][H_1] = \mathbf{V}[x][H_0][G_1].$$

By the same token, one can construct H_2 which is $\mathrm{Col}(\omega, \delta + 1)$-generic over $\mathbf{V}[x]$ such that

$$\mathbf{V}[x][H_2] = \mathbf{V}[x][H_1] = \mathbf{V}[x][H_0][G_1]$$

Finally, we get:

$$\mathbf{V}[G] = \mathbf{V}[G_0][G_1][G_2] = \mathbf{V}[x][H_0][G_1][G_2] = \mathbf{V}[x][H_2][G_2]$$

and since $\mathrm{Col}(\omega, \kappa) = \mathrm{Col}(\omega, \delta + 1) \times \mathrm{Col}(\omega, > \delta)$, one can check that $H_2 \cup G_2$ is $\mathrm{Col}(\omega, \kappa)$-generic over $\mathbf{V}[x]$.

□

Next, we explore a property of the Lévy Collapse which will allow us to determine the behaviour of the extension without knowing the generic filter.

Definition 2.2 *A partial order $\mathbb{P}$ is weakly homogeneous if for any $p, q \in \mathbb{P}$ there is an order automorphism χ of $\mathbb{P}$ such that $\chi(p) \parallel q$.*

Lemma 2.4

1. *If $\mathbb{P}$ is a weakly homogeneous forcing p.o. then for every formula $\varphi(\check{x}_1, \check{x}_2, \ldots, \check{x}_n)$ of the forcing language, where $x_1, \ldots, x_n \in \mathbf{V}$ we have either:*

$$\mathbb{1} \Vdash_{\mathbb{P}} \varphi(\check{x}_1, \check{x}_2, \ldots, \check{x}_n) \text{ or } \mathbb{1} \Vdash_{\mathbb{P}} \neg\varphi(\check{x}_1, \check{x}_2, \ldots, \check{x}_n)$$

2. $\mathrm{Col}(\lambda, \kappa)$ *is weakly homogeneous.*

Proof For the first part, suppose that there were $p, q \in \mathbb{P}$, such that $p \Vdash_{\mathbb{P}} \varphi(\check{x}_1, \check{x}_2, \ldots, \check{x}_n)$ and $q \Vdash_{\mathbb{P}} \neg\varphi(\check{x}_1, \check{x}_2, \ldots, \check{x}_n)$. Choose an automorphism χ of $\mathbb{P}$ such that $\chi(p) \parallel q$, which in particular is also a dense embedding. Then χ induces a bijection $\mathbb{P}$-names $\to$ $\mathbb{P}$-names, by inducting via $\chi_*(\{(\underset{\sim}{y}_\alpha, p_\alpha) : \alpha < \beta\}) = \{(\chi_*(\underset{\sim}{y}_\alpha), \chi(p_\alpha)) : \alpha < \beta\}$. Since χ fixes $\mathbf{0}$, it also fixes all $\mathbb{P}$-names $\check{x}$ with $x \in \mathbf{V}$. Furthermore, adapting the proof from [4][p.222] we also see that this bijection preserves the forcing relationship. Suppose $p \in G$, where G is $\mathbb{P}$-generic and:

$$p \Vdash_{\mathbb{P}} \varphi(\check{x}_1, \check{x}_2, \ldots, \check{x}_n).$$

Then by the forcing theorem $\mathbf{V}[G] \models \varphi(x_1, \ldots, x_n)$. Now since $\mathbf{V}[\chi(G)] = \mathbf{V}[G]$ by the dense embedding, by another application of the forcing theorem, there is $r \in \chi(G)$ such that

$$r \Vdash_{\mathbb{P}} \varphi(\check{x}_1, \check{x}_2, \ldots, \check{x}_n)$$

In particular we can choose this $r = \chi(p)$, since this reasoning can be done for any G containing p. Hence we have proven that:

$$\chi(p) \Vdash_{\mathbb{P}} \varphi(\check{x}_1, \check{x}_2, \ldots, \check{x}_n),$$

whih however contradicts the fact that $\chi(p) \parallel q$ and $q \Vdash_{\mathbb{P}} \neg\varphi(\check{x}_1, \check{x}_2, \ldots, \check{x}_n)$.

For the second part, we construct such a χ explicitly. Consider $p, q \in \mathrm{Col}(\lambda, \kappa)$. Then we have that $|\mathrm{dom}(p)|, |\mathrm{dom}(q)| < \lambda$ as before. Define a bijection $f : \lambda \to \lambda$ which fixes $\bigcup \mathrm{dom}(q)$ and maps $\bigcup \mathrm{dom}(p)$ into $\lambda \setminus \bigcup \mathrm{dom}(q)$, so that $\mathrm{dom}(f \circ p_\alpha) \cap \mathrm{dom}(q_\beta) = \emptyset$ for all $\alpha, \beta < \kappa$. Define the automorphism of $\mathrm{Col}(\lambda, \kappa)$ by $(\chi(p))_\alpha = f \circ p_\alpha$. We then clearly have $\chi(p) \parallel q$, since these two conditions have disjoint domains. □

The culmination of the work done so far is the following lemma, which allows us to reason about sequences of ordinals $x \in {}^\omega\mathbf{On} \cap \mathbf{V}[G]$ (where G is $\mathrm{Col}(\omega, \kappa)$-generic) by considering the smallest possible model in which it makes sense to reason about x, namely $\mathbf{V}[x]$.

Proposition 2.11 *Suppose that $\kappa > \omega$ is inaccessible and G is $\mathrm{Col}(\omega, \kappa)$-generic. Then for each formula $\varphi(v)$ there is a corresponding formula $\tilde{\varphi}(v)$ such that for any $x \in {}^\omega\mathbf{On} \cap \mathbf{V}[G]$ we have:*

$$\mathbf{V}[G] \vDash \varphi(x) \Leftrightarrow \mathbf{V}[x] \vDash \tilde{\varphi}(x)$$

Proof By the factoring lemma 2.3, there is a $\mathrm{Col}(\omega, \kappa)$-generic filter H over $\mathbf{V}[x]$ such that $\mathbf{V}[G] = \mathbf{V}[x][H]$. Now, using weak homogeneity of the Lévy collapse (lemma 2.4), we can already decide $\varphi(x)$ in $\mathbf{V}[x]$ as follows:

$$\mathbf{V}[G] = \mathbf{V}[x][H] \vDash \varphi(x) \Leftrightarrow \mathbf{V}[x] \vDash \ulcorner \Vdash_{\mathrm{Col}(\omega,\kappa)} \varphi(x) \urcorner =: \tilde{\varphi}(x).$$

As by weak homogeneity, it can already be determined in $\mathbf{V}[x]$, whether or not $\tilde{\varphi}(x)$ is satisfied, the proposition follows. □

2.4 Random and Cohen reals

Next, we will look at auxiliary forcing notions, which will provide characterizations of measurable sets and sets with the Baire property.

Consider the forcing partial orders on non-null and non-meagre sets respectively:

$$\mathcal{L}^\star = \{X \in \mathcal{L} : X \text{ non null}\}$$
$$\mathcal{B}^\star = \{X \in \mathcal{B} : X \text{ non null}\}$$

$$\mathcal{B}^\dagger = \{X \in \mathcal{B} : X \text{ not meager}\}$$

where $X \geq Y :\Leftrightarrow X \subset Y$. For now, let us focus on $\mathcal{L}^\star$. It has the following properties:

Proposition 2.12 (Properties of $\mathcal{L}^\star$)

1. *If $X, Y \in \mathcal{L}^\star$, then $X \perp Y \leftrightarrow X \cap Y$ is null.*
2. *$\mathcal{L}^\star$ has the ω_1-chain condition.*

Proof The first part follows from the fact, that every element bigger than both X and Y has to be a subset of $X \cap Y$. If however $X \cap Y \notin \mathcal{L}^\star$, then $X \perp Y$ and vice versa.

For the second part, assume we had an ω_1-anti chain in $\mathcal{L}^\star$. This corresponds to almost disjoint subsets of the real line by the first part. Since all of these however have non-zero measure, there are in particular ω_1 many with measure bigger than $\varepsilon > 0$ for ε small enough. Further refining, ω_1 many of those have at least half their measure contained in an interval $[-k, k]$. This however is impossible, since a bounded interval can only contain a finite number of almost disjoint sets with measure bounded from below by $\frac{\varepsilon}{2}$. □

We will now pass to the separative quotient of $\mathcal{L}^\star$ to get a better grasp of the forcing notion. Recall that the we can define an equivalence relation on $\mathcal{L}^\star$ by stipulating:

$$X \sim Y :\Leftrightarrow \forall Z(Z \parallel X \leftrightarrow Z \parallel Y)$$

In fact by the previous proposition, we see that $X \sim Y \Leftrightarrow X \Delta Y$ is null, i.e. two conditions are identical in the separative quotient if they are the same up to a null set. The same holds true for $\mathcal{B}^\star$. Since $\mathcal{L}$ and $\mathcal{B}$ only differ by null sets we have :

$$\mathcal{B}^\star / \sim \quad \cong \quad \mathcal{L}^\star / \sim$$

So forcing with $\mathcal{B}^\star, \mathcal{L}^\star$ or either of their separative quotients yield the same extensions, by proposition 2.4. Let us now determine what a form a generic object for $\mathcal{B}^\star$ (and hence for $\mathcal{L}^\star$) has.

Proposition 2.13 (Forcing with $\mathcal{B}^\star$) *Suppose that G is $\mathcal{B}^\star$-generic. Then there is a unique $x \in {}^\omega\omega \cap \mathbf{V}[G]$ such that for any closed code $c \in \mathbf{V}$:*

$$x \in A_c^{\mathbf{V}[G]} \Leftrightarrow A_c^{\mathbf{V}} \in G$$

and $\mathbf{V}[x] = \mathbf{V}[G]$.

Proof We will first show that certain subsets of $\mathcal{B}^\star$ are dense.

Claim *For every $n \in \omega$ the set:*

$$\mathcal{D}_n = \{C \in \mathcal{B} : C \textit{ closed } \wedge \exists k \in \omega(C \subset \{f \in {}^\omega\omega : f(n) = k\}\} \subset \mathcal{B}^\star$$

is dense.

Proof of claim Let $n \in \omega$ be fixed. Consider any non-null Borel set $A \in \mathcal{B}^\star$. Then by proposition 1.4 there are $O \subset {}^\omega\omega$ open and $C \subset {}^\omega\omega$ closed, such that $C \subset A \subset O$ and $\lambda(O \setminus C) < \frac{\lambda(A)}{2}$. So in particular, if $p \in \mathcal{B}^\star$ is any condition, we can find a closed $c \in \mathcal{B}^\star$ with $p \leq c$. Define $A_k := \{f \in {}^\omega\omega : f(n) = k\}$, which is a closed set, since it is the complement of finitely many cylinders $O(s)$ with $s \in \omega^\omega$. Then $c = \bigsqcup_{k \in \omega}(c \cap A_k)$ is a disjoint union of countably many closed sets. Hence at least one of the $c \cap A_k$ has non-zero measure and so $p \leq c \cap A_k \in \mathcal{D}_n$. ■

Claim *For every $A \in \mathcal{L}$ the set:*

$$\mathcal{E}_A = \{C \in \mathcal{B}^\star : C \textit{ is closed } \wedge (C \subset A \vee C \cap A = \emptyset)\} \subset \mathcal{B}^\star$$

is dense.

Proof of claim Let $B \in \mathcal{B}^\star$. If $\lambda(B \cap A) > 0$, then we can find $C \subset B \cap A$ closed with $\lambda(C) > 0$ by the previous remark. If not, then $\lambda(B \cap A^{\mathsf{c}}) > 0$, then choose $C \subset B \cap A^{\mathsf{c}}$ closed with $\lambda(C) > 0$. ■

In $\mathbf{V}[G]$, define a a real $x \in {}^{\omega}\omega$ by:

$$\{x\} = X = \bigcap \left\{ A_c^{\mathbf{V}[G]} : c \in \mathbf{V} \text{ closed code } \wedge A_c^{\mathbf{V}} \in G \right\}$$

This is well-defined for the following reasons. First, note that the intersection is over a system of closed sets (by definition of c closed) that has the finite intersection property (since G is a filter). By the first claim there exists for any $n \in \omega$ a $A_{c_n}^{\mathbf{V}} \in \mathcal{D}_n \cap G$, since G is $\mathcal{B}^\star$-generic, and $\mathcal{D}_n$ is dense. This implies that there is k_n such that for $x \in X$ we must have $x(n) = k_n$. Hence, there can be at most a single element in X. Consider that element $x \in {}^{\omega}\omega$ with $x(n) = k_n$ as described above, and let c be a closed code in $\mathbf{V}$ with $A_c^{\mathbf{V}} \in G$. Then by shrinking $A_c^{\mathbf{V}}$ and using claim 2, we get that there is $A_{c_n}^{\mathbf{V}} \geq A_c^{\mathbf{V}}$ such that $\emptyset \neq A_{c_n}^{\mathbf{V}} \subset O(x|_n)$. Hence by absoluteness there are $x_n \in A_c^{\mathbf{V}[G]}$ with $d(x, x_n) < \frac{1}{n}$, where d is the metric on the Baire space defined in the first chapter. However since $A_c^{\mathbf{V}[G]}$ is closed, and $x_n \xrightarrow{n \to \infty} x$ we must have $x \in A_c^{\mathbf{V}[G]}$. We can now conclude that $x \in X$.

Now suppose $c \in \mathbf{V}$ is a closed code. If $A_c^{\mathbf{V}} \in G$, then by definition $x \in A_c^{\mathbf{V}[G]}$. On the other hand, if $x \in A_c^{\mathbf{V}[G]}$, we want to conclude that $A_c^{\mathbf{V}} \in G$. Let $d \in \mathbf{V}$ be another closed code. Then clearly $\emptyset \neq \{x\} \subset A_d^{\mathbf{V}[G]} \cap \subset A_d^{\mathbf{V}[G]}$, and so by absoluteness we get that already in $A_d^{\mathbf{V}} \cap A_c^{\mathbf{V}} \neq \emptyset$. By genericity of G and the second claim, we can find $A_e^{\mathbf{V}} \in G \cap \mathcal{E}_{A_c^{\mathbf{V}}}$. However, since for every closed code $d \in \mathbf{V}$ we have $A_d^{\mathbf{V}} \cap A_c^{\mathbf{V}} \neq \emptyset$, we must have $A_e^{\mathbf{V}} \subset A_c^{\mathbf{V}}$ and hence $A_c^{\mathbf{V}} \in G$.

Finally, we can define G already in $\mathbf{V}[x]$ via:

$$G = \{p \in \mathcal{B}^\star : \exists \text{ closed code } c \in \mathbf{V}(x \in A_c^{\mathbf{V}[x]} \wedge A_c^{\mathbf{V}[x]} \subset p)\}$$

by using the fact that $x \in A_c^{\mathbf{V}[x]} \leftrightarrow x \in A_c^{\mathbf{V}[G]}$. This means that $G \in \mathbf{V}[x] \subset \mathbf{V}[G]$, so $\mathbf{V}[x] = \mathbf{V}[G]$. □

In other words, when we force with $\mathcal{B}^\star$ we add a unique real to our ground model, by prescribing in which decreasing sequence of Borel sets it should lie. This leads us to the following definition:

Definition 2.3 *Let M be an inner model. A real x in $\mathbf{V}$ is called random over M if there is a $(\mathcal{B}^\star)^M$-generic G over M and x is as in* 2.13*, i.e. $x \in M[G]$ and for every closed code $c \in M$:*

$$x \in A_c^{M[G]} \Leftrightarrow A_c^M \in G$$

and $M[G] = M[x]$.

Now there is another characterization of random reals:

Proposition 2.14 *Let $\mathbf{V}[G]$ be a forcing extension of $\mathbf{V}$. Then*

$$x \in {}^{\omega}\omega \cap \mathbf{V}[G] \textit{ is random over } \mathbf{V}$$
$$\Leftrightarrow$$
$$x \notin A_c \textit{ for any } c \in {}^{\omega}\omega \cap \mathbf{V} \textit{ which is a } G_\delta \textit{ code for a null set}$$

Hence we can alternatively think of random reals as new reals not part of any 'easily describable' (in the sense of G_δ) null set of the ground model.

Proof Suppose first that $x \in {}^\omega\omega \cap \mathbf{V}$ is random over M with a corresponding $(\mathcal{B}^\star)^M$-generic G and c is a G_δ code for a null set. Then:

$$M \vDash \ulcorner \mathcal{D} = \{C \in \mathcal{B}^\star : C \text{ closed } \wedge C \subset {}^\omega\omega \setminus A_c\} \text{ is dense in } \mathcal{B}^\star \urcorner$$

This is just because A_c^M is null, so that for any set $B \in (\mathcal{B}^\star)^M$ we have that $B \setminus A_c^M$ is not null, and by the above construction we can find a closed subset of non-zero measure. We can hence find a closed code $d \in M$ such that $A_d^M \in G \cap \mathcal{D}$. This however means that $A_c^M \cap A_d^M = \emptyset$. By the previous theorem, we get $x \in A_d^{M[x]}$, since $A_d^M \in G$. Hence by disjointness, we must have $x \notin A_c^{M[x]}$, so that $x \notin A_c^{\mathbf{V}}$.

On the other hand, suppose that $x \in A_c^{\mathbf{V}}$ whenever $c \in M$ is a G_δ code for a null set. We would like to conclude that x is random over M. In analogy to the construction of the generic filter from the random real, define in M:

$$G = \{p \in \mathcal{B}^\star : \exists \text{ closed code } c \in M(x \in A_c^{M[x]} \wedge A_c^{M[x]})\}$$

If we can prove that G is a $(\mathcal{B}^\star)^M$-generic filter, then we are done, since in this case G induces a random real, which is exactly equal to x. So suppose $\mathcal{D} \subset M$ is generic and argue in M.

Let W be a maximal anti-chain $W \subset \{C \in \mathcal{B}^\star : C \text{ closed } \wedge \exists B \in \mathcal{D}(C \subset B)\}$. Then by the ω_1-c.c. W is countable, so we can enumerate it: $W = \{A_{c_n} : n \in \omega\}$. Let c be the G_δ code for $\bigcap_{n\in\omega}({}^\omega\omega \setminus A_{c_n})$, which is in fact a null set. If it were not, then we could find a closed non-null subset $C \in \mathcal{B}^\star$ with the property that $C \cap A_{c_n} = \emptyset$, so that by density of $\mathcal{D}$ we can find another non-null subset $\tilde{C} \subset C \cap P$ with $P \in \mathcal{D}$. However $W \cup \{\tilde{C}\}$ would then be an anti-chain, which contradicts maximality. By hypothesis we have $x \notin A_c$, since c is a G_δ code for a null set, and hence there is $n \in \omega$ such that $x \in A_{c_n}$ and $P \in \mathcal{D}$ with $A_{c_n} \subset P$. Hence $A_{c_n} \in G \cap \mathcal{D}$ and so G is $(B^\star)^M$-generic. □

The preliminary work on the Lebesgue measure has now been done. Let us next outline the steps that need to be taken to adapt the treatment to the case of the Baire property. Unsurprisingly, let us consider $\mathcal{B}^\dagger$ and derive the analogues of the three previous propositions.

Proposition 2.15 (Properties of $\mathcal{B}^\dagger$)

1. *If $X, Y \in \mathcal{B}^\dagger$, then $X \perp Y \leftrightarrow X \cap Y$ is meagre.*
2. *The forcing p.o. $\mathcal{B}^\dagger$ has the ω_1-chain condition.*

Proof For the first part, every element bigger than both X and Y has to be a subset of $X \cap Y$, so if $X \cap Y$ is meagre, then $X \perp Y$ and vice versa.

For the second part, consider the separative quotient $\mathcal{B}^\dagger/\sim$ of $\mathcal{B}^\dagger$. We have that $X \sim Y \Leftrightarrow X \Delta Y$ is meagre. Since every Borel set has the Baire property, i.e. is open up to a meagre set, it follows that $\mathcal{B}^\dagger/\sim$ has a countable dense subset, namely the collection of all basic open sets $O(s)$ for $s \in \omega^{<\omega}$. Since every anti-chain consisting only of open sets can at most be countable, the same has to be true for $\mathcal{B}^\dagger/\sim$. Finally, since $\mathcal{B}^\dagger$ embeds densely into $\mathcal{B}^\dagger/\sim$, we have that $\mathcal{B}^\dagger$ also has the ω_1-c.c. □

Remark *For the last proof we relied on the fact that Borel sets are almost open. If instead we consider the p.o. given by all non-meagre sets:*

$$\mathcal{M}^\dagger = \{X \subset {}^\omega\omega : X \text{ is not meagre}\}$$

then the proof does not work, and in fact cannot work, as one can construct by transfinite induction a collection of $\mathfrak{c}$ disjoint non-meagre sets. Start by enumerating the $\mathfrak{c}$ meagre subsets of $\mathbb{R}$, say $(N_\alpha)_{\alpha<\mathfrak{c}}$. Choose a bijection $b : \mathfrak{c} \to \mathfrak{c} \times \mathfrak{c}$. Now build up a collection $(S_\beta)_{\beta<\mathfrak{c}}$ of disjoint sets by adding at step α an element $\mathbb{R} \setminus N_\gamma$ to S_β, where $f(\alpha) = (\beta, \gamma)$. Since $X \setminus M_\gamma$ has cardinality $\mathfrak{c}$, we can choose these elements to be distinct. Hence we end up with a collection of $\mathfrak{c}$ disjoint sets, none of which is entirely contained in a meagre set, hence all of them have to be non-meagre. Hence in $\mathcal{M}^\dagger$ there are anti-chains of size $\mathfrak{c}$. However it necessarily has to contain non-Borel sets.

In analogy to the Lebesgue case we can characterize $\mathcal{B}^\dagger$ in terms of adding a single real, just by tweaking the formulation slightly.

Proposition 2.16 (Forcing with $\mathcal{B}^\dagger$) *Suppose that G is $\mathcal{B}^\dagger$-generic. Then there is a unique real $x \in {}^\omega\omega \cap \mathbf{V}[G]$ such that for any G_δ code $c \in \mathbf{V}$:*

$$x \in A_c^{\mathbf{V}[G]} \Leftrightarrow A_c^{\mathbf{V}} \in G$$

and $\mathbf{V}[x] = \mathbf{V}[G]$.

Proof Consider the analogues of the claims in the proof of 2.13.

Claim *For every $n \in \omega$ the set:*

$$\mathcal{D}_n = \{C \in \mathcal{B}^\dagger : C \text{ is } G_\delta \wedge \exists k \in \omega(C \subset \{f \in {}^\omega\omega : f(n) = k\}\} \subset \mathcal{B}^\dagger$$

is dense.

Proof of claim By proposition 1.8, we can approximate from inside every non-meagre Borel set $A \in \mathcal{B}^\dagger$ by a non-meagre G_δ set $G \subset A$. Define $A_k = \{f \in {}^\omega\omega : f(n) = k\}$, which are all G_δ. Hence $G = \bigcup_{k\in\omega}(A_k \cap G)$ is non-meagre, hence at least one of the $A_k \cap G$ is a non-meagre G_δ set in $\mathcal{D}_n$. ■

Claim *For every $A \in \mathcal{B}$ the set:*

$$\mathcal{E}_A = \{C \in \mathcal{B}^\dagger : C \text{ is } G_\delta \wedge (C \subset A \vee C \cap A = \emptyset)\} \subset \mathcal{B}^\dagger$$

is dense.

Proof of claim The union of two meagre sets is meagre, hence if $B \in \mathcal{B}^{\dagger}$, then either $A \cap B$ or $A^{\mathsf{c}} \cap B$ has to be non-meagre and must contain a G_δ subset by proposition 1.8. ∎

In $\mathbf{V}[G]$, we consider:

$$X = \bigcap \left\{ A_c^{\mathbf{V}[G]} : c \in \mathbf{V} \text{ is a } G_\delta \text{ code } \wedge A_c^{\mathbf{V}} \in G \right\}$$

If we consider the same intersection in the ground model:

$$\emptyset = \bigcap \left\{ A_c^{\mathbf{V}} : c \in \mathbf{V} \text{ is a } G_\delta \text{ code } \wedge A_c^{\mathbf{V}} \in G \right\}$$

we clearly get that it's empty by genericity, since the complement of a generic filter is dense. Hence any elements of X must be new reals not present in $\mathbf{V}$.

Consider $A_c^{\mathbf{V}[G]}$ for some G_δ code $c \in {}^{\omega}\omega \cap \mathbf{V}$ and $A_c^{\mathbf{V}} \in G$. Given $n \in \omega$, by density of the $\mathcal{D}_n$ we can find a condition $p \in G$ such that $p \geq A_c^{\mathbf{V}}$ and $p \geq q_i \in \mathcal{D}_i$ for some q_i and for all $i < n$. By density of $\mathcal{E}_p$ we can further find $q \geq p$ with $q \in G$ being G_δ. Hence by passing to a subset, we may assume that $A_c^{\mathbf{V}[G]} \subset O(x|_n)$ for every fixed $N \in \omega$, where x is as constructed through claim 1 (in analogy to the Lebesgue case). It is clear from this that x is the only possible element of X.

(Starting here this is just a guess of what a proof of $x \in X$ might look like) Starting from $A_c^{\mathbf{V}}$, construct now an increasing sequence of conditions in G of the form $A_c^{\mathbf{V}} = p_0 \leq p_1 \leq p_2 \leq \dots$ such that $p_n \subset O(x|_n)$. This shows that we can construct $x_n \in A_c^{\mathbf{V}[G]}$ with $d(x, x_n) < \frac{1}{n}$. Suppose now that $x \notin A_c^{\mathbf{V}[G]} = U \cap \bigcap_{k \in \omega} O_k$, where O_k and U are open sets. We may assume that $x \notin U$, which can be written as $U = \bigcup_{i \in \omega} O(s_i)$. Here $x \notin U$ means that s_i is not a prefix of x for any $i \in \omega$. Now here it should be possible to reach a contradiction using the fact that c is part of the ground model, and hence cannot explicitly exclude x. Consider:

$$S = \{ s \in {}^{<\omega}\omega : \forall i \in \omega : (s \text{ is not a prefix of } s_i) \} \subset {}^{<\omega}\omega$$

Recall that c allows to recover all the different s_i and hence this set is already in $\mathbf{V}$. The elements of this set are all the prefixes that are not explicitly included in U. Suppose some prefix of x was in S. Then $U \cap O(x|_n) = \emptyset$, which is a contradiction. Hence every $x|_n$ appears as the prefix of some s_i. Define:

$$F = \{ f \in {}^{\omega}\omega : \forall n \in \omega \exists i \in \omega (f|_n \text{ is a prefix of } s_i) \wedge \forall i \in \omega (s_i \text{ is not a prefix of } f) \}$$

For this set we clearly have $x \in F^{\mathbf{V}[G]}$ and $x \notin F^{\mathbf{V}}$. Now we would need to reach a contradiction somehow. (End of guessing) The intersection contains x since $x \in A_c^{\mathbf{V}[G]}$ for every G_δ code c from the ground universe. The equivalence and $\mathbf{V}[x] = \mathbf{V}[G]$ is identical to the Lebesgue case if "closed" is replaces by "G_δ". □

However, in this case we do not have to define a new type of generic real, since as the next proposition demonstrates forcing with $\mathcal{B}^\dagger$ adds a Cohen real.

Proposition 2.17 *Forcing with $\mathcal{B}^\dagger$ coincides with forcing with Cohen forcing $\mathbb{C}$.*

Proof The countable dense subset of $\mathcal{B}^\dagger / \sim$ given by the representatives of the basic open sets $O(s)$ is a forcing notion that is separable, countable and adds a generic real (since $\mathcal{B}^\dagger$ does by the previous proposition). Hence it satisfies the conditions for proposition 2.10 with $\alpha = \omega$, and we get an injective dense embedding ι from a dense subset of $\mathbb{C}$. We can summarize the situation by the following diagram of forcing notions and dense embeddings:

$$\begin{array}{ccccc} \mathbb{C} = \mathbb{P}^\omega_\omega & & \mathcal{B}^\dagger / \sim & \xleftarrow[p]{\quad\quad} & \mathcal{B}^\dagger \\ \uparrow & & \uparrow & & \\ D & \xrightarrow{\ \iota\ } & \{O(s)^\sim : s \in \omega^{<\omega}\} & & \end{array}$$

Here the vertical maps are given by the inclusion of dense subsets and p is the projection onto the separative quotient. It follows that $\mathcal{B}^\dagger$ is equivalent to $\mathbb{C}$. □

Proposition 2.18 *Let $\mathbf{V}[G]$ be a forcing extension of $\mathbf{V}$. Then*

$$x \in {}^\omega\omega \cap \mathbf{V}[G] \text{ is Cohen over } \mathbf{V}$$
$$\Leftrightarrow$$
$$x \notin A_c \text{ for any } c \in {}^\omega\omega \cap \mathbf{V} \text{ which is a } F_{\sigma\delta} \text{ code for a meagre set}$$

Proof The proof is identical to the proof of 2.14, if "G_δ" is replaced by "$F_{\delta\sigma}$" and "null" by "meagre". Of course we need to utilize the properties of $\mathcal{B}^\dagger$ instead of $\mathcal{B}^\star$, however they exactly parallel each other. □

2.5 Proof of Solovay's theorem

It is now finally time to embark on the proof of Solovay's theorem 2.1. The strategy is to first force with $\mathrm{Col}(\omega, \kappa)$ to obtain an extension $\mathbf{V}[G]$ with a lot of new reals that can be reasoned about from within $\mathbf{V}$, and then pick out an inner model of especially nice sets.

Definition 2.4 *A set X is said to be ${}^\omega\mathbf{On}$-definable if there is $a \in {}^\omega\mathbf{On}$ and a formula $\varphi(v_1, v_2)$ such that*

$$y \in X \leftrightarrow \varphi[y, a]$$

This notion is formalisable in **ZFC** *via the reflection principle as follows:*

$$\exists\alpha\exists a\exists\varphi(a \in {}^\omega\mathbf{On} \cap \mathbf{V}_\alpha \wedge \forall y(y \in X \leftrightarrow y \in \mathbf{V}_\alpha \wedge \mathbf{V}_\alpha \vDash \varphi[a, y]))$$

The key step in constructing the Solovay model is the following theorem:

Theorem 2.4 *Suppose that κ is an inaccessible cardinal and let G be* $\mathrm{Col}(\omega,\kappa)$*-generic. Then every* ${}^{\omega}\mathbf{On}$*-definable set of reals in* $\mathbf{V}[G]$ *is Lebesgue measurable, has the property of Baire, and has the perfect set property.*

Proof A consequence of the properties of the Lévy collapse we have:

"For any $a \in {}^{\omega}\mathbf{On}$ the set ${}^{\omega}\omega \cap \mathbf{V}[a]$ is countable as a subset of $\mathbf{V}[G]$"

By proposition 2.2, we can find $\delta < \kappa$ and $G_\delta = \pi_{<\delta}(G)$ such that $a \in \mathbf{V}[G_\delta]$ and G_δ is $\mathrm{Col}(\omega,\delta)$ generic. It follows that $\mathbf{V}[a] \subset \mathbf{V}[G_\delta]$. Since by inaccessibility of κ we have $|\mathrm{Col}(\omega,\delta)| \leq \delta 2^{\delta} < \kappa$, we get by proposition 2.5 that κ is still inaccessible in $\mathbf{V}[G_\delta]$, hence a fortiori in $\mathbf{V}[a]$. This means in particular that $|{}^{\omega}\omega \cap \mathbf{V}[a]| < \kappa$ in $\mathbf{V}[a]$. Since however we have that $\omega_1^{\mathbf{V}[G]} = \kappa$ by the definition of $\mathrm{Col}(\omega,\kappa)$, we have that ${}^{\omega}\omega \cap \mathbf{V}[a]$ must be countable in $\mathbf{V}[G]$.

From now on argue in $\mathbf{V}[G]$. Suppose that $A \in {}^{\omega}\omega$ is ${}^{\omega}\mathbf{On}$-definable. Then there is $\varphi(v_1, v_2)$ and $a \in {}^{\omega}\mathbf{On}$ such that:

$$x \in A \Leftrightarrow \varphi(a, x)$$

However, using a two-formula version of the key fact 2.11, we can decide $\varphi(a, x)$ already in $\mathbf{V}[a][x]$. There is a formula $\psi(v_1, v_2)$ such that:

$$x \in A \Leftrightarrow \mathbf{V}[a][x] \vDash \psi(a, x)$$

Let us first verify that A is Lebesgue measurable.

Claim *The following is a null set:*

$$Z = \{x \in {}^{\omega}\omega : x \textit{ not random over } \mathbf{V}[a]\}$$

Proof of claim Recall that x is random over $\mathbf{V}[a]$ iff $x \notin A_c^{\mathbf{V}[a]}$ for any G_δ code $c \in \mathbf{V}[a]$ for a null set. Hence

$$Z = \bigcup \{A_c : c \in \mathbf{V}[a] \text{ is a } G_\delta \text{ code for a null set}\}$$

Now since ${}^{\omega}\omega \cap \mathbf{V}[a]$ is countable, the set Z is a countable union of null sets, hence itself null. ∎

To prove A's measurability, by proposition 1.2 we need to find a Borel set B such that $A \Delta B$ is a null set. In particular, by the previous claim if we can restrict ourselves to finding $B \in \mathcal{B}$ such that $A \Delta B \subset Z$ consists only of non random reals, since every subset of a null set is a null set as well. We can in fact construct such a B explicitly.

Consider the $\mathcal{B}^{\star}$ forcing in $\mathbf{V}[a]$, and let the name of the canonical random real be $\underset{\cdot}{r}$. Let $W \in \mathbf{V}[a]$ be a maximal anti-chain of closed sets deciding $\psi(\underset{\cdot}{a}, \underset{\cdot}{r})$, i.e. deciding if $\underset{\cdot}{r}[G] \in A$ by the definition of ψ. Suppose now x is random over $\mathbf{V}[a]$ and H is the corresponding generic filter. Then:

$$x \in A \Leftrightarrow x \in B = \bigcup \{A_c : A_c^{\mathbf{V}[a]} \in W \wedge A_c^{\mathbf{V}[a]} \Vdash_{\mathcal{B}^\star \cap \mathbf{V}[a]} \psi(\underset{\cdot}{a}, \underset{\cdot}{r})\}.$$

Let us prove this equivalence. If $x \in A$, then $\mathbf{V}[a][x] \models \psi(a, x)$, and hence by the forcing theorem there is a condition $q \in H$ such that $q \Vdash \psi(\underset{\cdot}{a}, \underset{\cdot}{r})$. However, since W is a maximal anti-chain deciding exactly this formula, there is $q \geq A_c^{\mathbf{V}[a]} \in W$ such that $A_c^{\mathbf{V}[a]} \Vdash \psi(\underset{\cdot}{a}, \underset{\cdot}{r})$. But then $A_c^{\mathbf{V}[a]} \in H$ and hence by definition of the random real $x \in A_c$, so $x \in B$. If $x \in A_c \subset B$, then by proposition 2.13 we have $A_c^{\mathbf{V}[a]} \in H$. Since furthermore $A_c^{\mathbf{V}[a]} \Vdash \psi(\underset{\cdot}{a}, \underset{\cdot}{r})$ we conclude that $\mathbf{V}[a][x] \models x \in A$, hence also $x \in A$.

By the ω_1-c.c. of $\mathcal{B}^\star$ we get that W is countable. It follows that B is a countable union of Borel sets, hence Borel itself. Finally, by the equivalence above, $A \Delta B$ consists only of non-random reals, which means that A is Lebesgue measurable.

The treatment of the Baire property is very similar to the Lebesgue Measurability. A careful inspection of the proof will show that replacing $\mathcal{B}^\star$ with $\mathcal{B}^\dagger$, "null" by "meagre" and using the analogues of the propositions for the Baire property, with just a few aesthetic adjustements the proof goes through.

Finally, let us verify the perfect set property of A, which we can assume to be uncountable. Since ${}^\omega\omega \cap \mathbf{V}[a]$ is countable there exists $y \in A \setminus \mathbf{V}[a]$. By the factoring lemma 2.3, there is a $\mathrm{Col}(\lambda, \kappa)$-generic H over $\mathbf{V}[a]$ such that $\mathbf{V}[a][H] = \mathbf{V}[G]$. Since $y \in {}^\omega\omega$, we can apply 2.2 to get a forcing notion $\mathbb{Q} = \mathrm{Col}(\omega, \delta) \in \mathbf{V}[a]$ with $|\mathbb{Q}| < \kappa$ and $y \in \mathbf{V}[a][\pi_\alpha(H)]$.

Let $\underset{\sim}{y}$ be a name for y in $\mathbf{V}[a]$. By the forcing theorem there is a condition $p \in H$ such that:

$$p \Vdash_{\mathbb{Q}} \underset{\sim}{y} \in {}^\omega\omega \setminus \mathbf{V}[\underset{\cdot}{a}] \wedge \mathbf{V}[\underset{\cdot}{a}][\underset{\sim}{y}] \models \psi(\underset{\cdot}{a}, \underset{\sim}{y})$$

We have that $\mathcal{P}^{\mathbf{V}[a]}(Q)$ is countable in $\mathbf{V}[G]$, since $2^{|Q|} < \kappa$ in $\mathbf{V}[a]$ by inaccessibility of κ. So in $\mathbf{V}[G]$ we can enumerate the dense subsets of Q which are already present in $\mathbf{V}[a]$ as $\{D_n : n \in \omega\}$. Assign now to each $t \in {}^{<\omega}2$ a condition $p_t \in Q$ and a $u_t \in {}^{<\omega}\omega$ as follows. Choose first $p_\emptyset \geq p$ such that $p_\emptyset \in D_0$. Given $p_t \geq p$ with $t : n \to 2$, then since $p \Vdash_{\mathbb{Q}} \underset{\sim}{y} \notin \mathbf{V}[\underset{\cdot}{a}]$ there must be a least $k \in \omega$ such that p_t does not decide a value for $\underset{\sim}{y}(k)$. Let u_t satisfy $p_t \Vdash_{\mathbb{Q}} \underset{\sim}{y}|_k = \underset{\cdot}{u}_t$. Choose now $p_{t\frown 0}$ and $p_{t\frown 1}$ in D_{n+1} so that they decide different values for $\underset{\sim}{y}(k)$.

For $x \in {}^\omega 2$ set

$$G_x := \{q \in Q : \exists t (p_t \geq q \wedge t \subset x)\}$$

By construction, G_x is a $\mathbb{Q}$-generic filter over $\mathbf{V}[a]$. Set:

$$C = \{\underset{\sim}{y}[G_x] : x \in {}^\omega 2\} \subset {}^\omega\omega \cap \mathbf{V}[G]$$

Then $C \subset A$, since $p \in G_x$ and $p \Vdash_{\mathbb{Q}} \mathbf{V}[\dot{a}][\underset{\sim}{y}] \models \psi(\dot{a}, \underset{\sim}{y})$. Furthermore, we have for $x \in {}^{\omega}2$ that:

$$\{\underset{\sim}{y}[G_x]\} = \bigcap_{n\in\omega} O(u_{x|_n})$$

by construction. This implies that:

$$C = \bigcap_{n\in\omega} \bigcup_{t\in{}^{<\omega}2} \{O(u_t) : |t| = n\}$$

The set C does not contain isolated points, as the corresponding tree

$$T_C = \{u_t : t \in {}^{<\omega}2\}$$

has been constructed to be splitting indefinitely, i.e. perfect. Finally, as an intersection of closed sets (notice that the $O(s)$ are both open and closed sets), it is closed, and hence perfect. □

We are now finally able to proof the main theorem of this section. The main idea is to look at the inner model $L(\mathbb{R}) \subset \mathbf{V}[G]$, i.e. the constructible closure of the reals.

Proposition 2.19 *The axiom of dependent choice* **DC** *is satisfied in* $L(\mathbb{R})$.

Proof Consider a surjective map $\Phi : \mathbf{On} \times \mathbb{R} \to L(\mathbb{R})$ such that $\Phi|_{\alpha\times\mathbb{R}} \in L(\mathbb{R})$ as we introduced when defining the inner model $L(\mathbb{R})$ in section 2.1.

Let $X, R \in L(\mathbb{R})$ be given with $X \neq \emptyset$ and $R \subset X \times X$, such that the following is satisfied:

$$\forall x \in X \exists y \in X(\langle x, y\rangle \in R)$$

We want to show that there is $g \in {}^{\omega}X \cap L(\mathbb{R})$, called a *choice function for* **DC**, such that

$$\forall n \in \omega(\langle g(n), g(n+1)\rangle \in R)$$

Since in $\mathbf{V}$ we assume **AC**, which implies **DC**, we can immediately show that $\exists f \in {}^{\omega}X \cap \mathbf{V}$ which is satisfies the above relation. Choose $\gamma \in \mathbf{On}$ such that $X, R \in \Phi[\gamma \times \mathbb{R}]$. Furthermore, for each $n \in \omega$ we have $f(n) \in L(\mathbb{R})$, hence, there is $\delta_n \in \mathbf{On}$ such that $f(n) \in \Phi[\delta_n \times \mathbb{R}]$. By setting $\delta = \gamma \cup \bigcup_{n\in\omega} \delta_n$ we get that $\mathrm{ran}(f) \subset \Phi[\delta \times \mathbb{R}]$. Now, by $\mathbf{AC}_\omega$ in $\mathbf{V}$, construct $\langle a_n \in \mathbb{R} : n \in \omega\rangle$ such that

$$\forall n \in \omega \exists \xi_n < \delta(\Phi(\xi_n, a_n) = f(n))$$

Define the relation $\prec$ on $\delta \times \omega$ by:

$$(\eta, i) \prec (\zeta, j) \Leftrightarrow i = j + 1 \wedge \langle \Phi(\zeta, a_j), \Phi(\eta, a_i)\rangle \in R$$

Since countable sequences of reals can be coded by a single real, we have that $\langle a_n \in \mathbb{R} : n \in \omega\rangle \in L(\mathbb{R})$. Furthermore, the we have that $\Phi|_{\delta\times\mathbb{R}} \in L(\mathbb{R})$, and since $\prec$ is definable from those two objects, we get $\prec \in L(\mathbb{R})$

Next, we will prove that $\prec$ is ill-founded in a model iff there is a choice function present in that model. In $\mathbf{V}$, the existence of f then implies that $\prec$ is ill-founded, and in $L(\mathbb{R})$ the ill-foundedness will then imply the existence of $g \in {}^{\omega}X \cap L(\mathbb{R})$ as above, and hence conclude the proof. The link is made by $\ulcorner\prec$ is well-founded$\urcorner$ being a Δ^0_1-formula and hence absolute for transitive models of **ZF**, see 2.3.

Let f be a choice function. Then consider $E = \{\langle \xi_n, n\rangle : n \in \omega\} \subset \delta \times \omega$ as defined above. Since $f(n) = \Phi(\xi_n, a_n)$ we have that:

$$\langle \xi_0, 0\rangle \succ \langle \xi_1, 1\rangle \succ \dots,$$

so that E does not have a minimum and $\prec$ is ill-founded.

On the other hand, if $\prec$ is ill-founded, let $E \subset \delta \times \omega$ be a subset without $\prec$-minimal element. Then starting with $\langle \eta, i_0\rangle \in E$ define $\langle \eta_n : n \in \omega\rangle$ by setting $\eta_0 = \eta$ and choosing η_n minimal among all ordinals such that $\langle \Phi(\eta_n, a_{i_0+n}), \Phi(\eta_{n+1}, a_{i_0+n+1})\rangle \in R$. This construction can be done in $L(\mathbb{R})$, hence we can construct $g \in {}^{\omega}X \cap L(\mathbb{R})$ such that $g(n) = \Phi(\eta_n, a_{i_0+n})$. By unwinding the definition of $\prec$ we get that g is the desired choice function. □

Now we are finally ready to prove the main theorem 2.1.

Proof (Solovay's theorem) We consider $L(\mathbb{R}) \subset \mathbf{V}[G]$ and argue in $\mathbf{V}[G]$. Recall that $L(\mathbb{R})$ is defined to satisfiy ${}^{\omega}\omega \cap L(\mathbb{R}) = {}^{\omega}\omega$. The basic open sets $O(s)$ for $s \in {}^{<\omega}\omega$ are all in $L(\mathbb{R})$, and hence every Borel set is as well, since we can perform its construction inside $L(\mathbb{R})$ given the corresponding code.

Suppose now that $A \subset {}^{\omega}\omega$ with $A \in L(\mathbb{R})$. Then A is ${}^{\omega}\mathbf{On}$-definable, since taking finitely many arbitrary parameters can be model by countably many **On**-parameters.

Hence there is a Borel set $B \subset {}^{\omega}\omega$ with $B \in \mathbf{V}[G]$ such that $A\Delta B$ is null, since A is measurable in $\mathbf{V}[G]$. Hence there is a sequence of open sets O_n such that $\lambda(O_n) < \frac{1}{n}$ and $A\Delta B \subset O_n$. By absoluteness of the Lebesgue measure (See 2.9), and since O_n are also present in $L(\mathbb{R})$, the same holds true in $L(\mathbb{R})$. Hence $A\Delta B \subset \bigcap n \in \omega O_n$, where the set on the right-hand side is an $L(\mathbb{R})$ null set, and hence $A\Delta B$ is a null set in $L(\mathbb{R})$ as well. So A is measurable in $L(\mathbb{R})$.

Similarly there is a Borel set $B \subset {}^{\omega}\omega$ with $B \in \mathbf{V}[G]$ such that $A\Delta B$ is meagre, since A has the Baire property in $\mathbf{V}[G]$. As in the proof of 1.8, we can assume that $A\Delta B \subset F$ where F is the countable union of closed nowhere dense sets, hence F_δ. By proposition 2.8 we find that $A\Delta B \subset F$ in $L(\mathbb{R})$ as well, hence A has the property of Baire in the sense of $L(\mathbb{R})$.

For the perfect set property, we can assume that A is uncountable in $L(\mathbb{R})$. Suppose it was countable in $\mathbf{V}[G]$. In $\mathbf{V}[G]$, every set can be well-ordered by **AC**, hence A corresponds to a countable ordinal $\alpha < \omega_1$. Now by lemma 5.10 of [5, p.109], since α is countable, we can find a further order-isomorphism $h : \alpha \to Q \subset \mathbb{Q}$ to a subset of the rationals. However now, since subsets of the rationals can be identified with reals, this subset is already present in $L(\mathbb{R})$. Hence A can be identified with Q in $L(\mathbb{R})$ and hence is countable, contradiction. Hence A is uncountable in $\mathbf{V}[G]$. Thus A has a perfect subset C in $\mathbf{V}[G]$. This perfect subset is also present in $L(\mathbb{R})$ by its definition, and perfect in the sense of $L(\mathbb{R})$ by noting the tree corresponding to C is absolute and closedness of C is maintained in $L(\mathbb{R})$.

Finally $L(\mathbb{R})$ satisfies **DC** by the previous proposition. □

Chapter 3

Conclusion

In chapter 1, we discussed that Borel sets are Lebesgue measurable and have the property of Baire and that closed sets have the perfect set property. We furthermore showed that under **AC** it is possible to construct counterexamples for each of these three properties.

In chapter 2, we constructed the model of Solovay in which **DC** holds and every subset of the real line is measurable, has the property of Baire and has the perfect set property. Thereby we showed that **AC** is indeed necessary for the construction of the pathological sets from chapter 1, as the weaker **DC** does not imply their existence. However we assumed the existence of an inaccessible cardinal for the proof.

In this chapter we explore further results, expanding on and completing the picture painted by the last two chapters. We begin by extending the Borel hierarchy.

An *analytic* set or Σ^1_1 set is a subset of the reals which is the continuous image of a Borel set. The *coanalytic* or Π^1_1 sets are the complements of analytic sets. For any natural number $n \geq 1$ we define a set $A \subset {}^\omega\omega$ to be:

- A Σ^1_{n+1} set if there is a Π^1_n set $C \subset {}^\omega\omega \times {}^\omega\omega$ such that $A = \pi_1(C)$ is the projection of C onto the first coordinate.
- A Π^1_{n+1} set if it is the complement of a Σ^1_{n+1}-set.

Furthermore for $n \geq 1$ we define $\Delta^1_n = \Pi^1_n \cap \Sigma^1_n$.

We call this collection of increasingly complex sets the *projective hierarchy* and the collection:

$$\mathcal{P} = \bigcup_{n \geq 1} \Sigma^1_n = \bigcup_{n \geq 1} \Pi^1_n$$

the *projective sets*. Now we are in a position to expand on the first chapter. Indeed, there is no need for additional assumptions to show that the three regularity properties extend into the first projective level.

Theorem 3.1 (Properties of analytic sets)

1. *Every analytic set is Lebesgue measurable.*
2. *Every analytic set has the Baire property.*
3. *Every analytic set has the perfect set property.*

For the proof of the main theorem 2.1, we had to assume the consistency with **ZFC** of the existence of an inaccessible cardinal. The natural question to ask is then if it is really necessary to assume such an additional hypothesis. Shelah settled this question for Lebesgue measurability and the property of Baire in his paper ”Can you take Solovay’s inaccessible away?” [6]. More precisely he showed:

Theorem 3.2 (Shelah) *If every Σ^1_3 set of reals is Lebesgue measurable, then ω_1 is an inaccessible cardinal in L.*

This shows that if Lebesgue measurability is guaranteed for all Σ^1_3-sets, then the existence of inaccessible cardinals is consistent with **ZF**. Furthermore he showed that Σ^1_3 is optimal in this regard, as:

Theorem 3.3 (Shelah) *Every universe of set theory admits a generic extension in which every Δ^1_3-set of reals is measurable.*

In other words the statements ”There is an inaccessible cardinal” and ”Every set is Lebesgue measurable” are equiconsistent over **ZF**.

So far, Lebesgue measurability and the Property of Baire have been treated in parallel, with similar methods and results. There are however also clear difference between them, as the inaccessible cardinal needed in the Lebesgue case is not needed in the case of the Baire property. More explicitly:

Theorem 3.4 (Shelah) *Every universe of set theory satisfying the continuum hypothesis admits a generic extension in which every set definable in the extension with a real and an ordinal parameter has the property of Baire.*

So an analogous construction as in the proof of the main theorem leads to an inner model in which every set of reals has the property of Baire, but without the requirement of an inaccessible cardinal.

Finally, considering the perfect set property, Specker showed in [7] that if every set of reals has the prefect set property, then ω_1 is a strong limit cardinal in L. Hence, combining this with the main theorem 2.1, we get that ”Every subset of the reals has the perfect set property” is equiconsistent over **ZF** with ”There is an inaccessible cardinal”.

We have now completed our investigations into the model of Solovay and the benefits obtained by abandoning **AC** in favour of the weaker **DC**. There are however other axioms worth investigating such as the *axioms of determinacy*.

Fix a subset of the Baire space $A \subset \mathcal{N}$ and consider the countably infinite two-player game in which each player alternately picks a natural number. In this manner, a sequence $\langle a_n : n \in \omega \rangle \in \mathcal{N}$ is generated. We say that the first player wins, if $\langle a_n : n \in \omega \rangle \in A$, else the second play wins. Consider the following statements:

BD : "The game described above is determined if $A \in \mathcal{B}$"

PD : "The game described above is determined if $A \in \mathcal{P}$"

AD : "The game described above is determined if A is arbitrary"

The first statement, Borel determinacy **BD** is in fact a theorem of **ZFC**. The second statement is a strengthening of **BD** and is called the *axiom of projective determinacy*. It cannot be proved in **ZFC**, but it follows from certain large cardinal hypothesis and has the following consequence:

Theorem 3.5 (PD and regularity) *The following are implied by* **PD***:*

1. *Every projective set is Lebesgue measurable.*
2. *Every projective set has the Baire property.*
3. *Every projective set has the perfect set property.*

For a proof of this theorem see [8].

The last statement is called the *axiom of determinacy* and it implies that all sets of reals are Lebesgue measurable, have the property of Baire and have the perfect set property. Hence it contradicts **AC**. Due to its pleasing consequences for the reals, it is a possible axiom for set theory, an alternative to **AC**.

To wrap up, we have seen that the regularity properties of the real line are intimately connected with strong assumptions, such as the existence of inaccessible cardinals or the axioms of determinacy. The analytical sets are unconditionally regular, and having all sets regular implies the failing of **AC**. In between these two extremes certain regularity properties can be proved using strong assumptions, and in turn having certain regularity properties implies strong assumptions. In this sense, the behaviour of the reals has implications on the rest of the set theoretic universe, and the impact of strong assumptions on the reals can be used as a gauge for their strength.

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List of Symbols

$\varphi, \psi, \ldots$	First order formulae
ZF	Zermelo-Frankel Set theory
ZFC	**ZF** together with the axiom of choice
V	The set-theoretic universe
$M \vDash \varphi$	M is a model of φ
$\mathbb{P} = (P, \leq), \mathbb{Q}, \ldots$	Forcing partial orders
$p \leq q$, for $p, q \in \mathbb{P}$	q is stronger than p
$\mathrm{Fn}(X, Y)$	Partial functions from X to Y
$G, H \subset \mathbb{P}$	Generic filters
$p \parallel q$	p is compatible with q
$p \perp q$	p is incompatible with q
$\mathbf{0} \in \mathbb{P}$	Minimal element of a partial order
$p \Vdash_{\mathbb{P}} \varphi$	p forces φ
$\dot{x}$	canonical name for set
$\dot{G}$	canonical name for generic filter
$\tilde{x}$	name for unspecific object
$\mathrm{Col}(\lambda, \kappa)$	Lévy Collapse
$\mathcal{N}$	Baire Space
$\mathcal{B}$	Borel σ-algebra
$\mathcal{L}$	Lebesgue measurable sets

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