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Instanton Floer homology and the spectral flow

Master Thesis

Gilles Englebert

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Advisor: Prof. W. Merry

Department of Mathematics, ETH Zürich

Prelude

Floer homology is in essence the extension of the usual Morse homology of closed finite-dimensional manifolds to certain infinite-dimensional situations, where the naive Morse index of a critical point is not necessarily finite, and compactness becomes a subtle issue. In these notes we focus on the **instanton Floer homology** $\mathrm{HF}^\bullet(N)$ of a homology three-sphere N, which is the Morse homology of the **Chern-Simons functional** CS. Defined on the infinite dimensional space of gauge equivalence classes of $\mathrm{SU}(2)$-connections on N and being circle-valued, this functional brings in further features not present in classical Morse homology. For one, due to the presence of reducible connections, the gauge equivalence classes of connections only form a manifold away from a collection of singular points. Furthermore, since CS is circle-valued it can occur that gradient flow lines form loops, making the trajectory spaces more subtle to describe.

In order to address the difficulties arising from the definition of the Morse index, we will introduce the **spectral flow** of a family of self-adjoint operators between separable Hilbert spaces, a tool common to all Floer-type theories. Using it, we will be able to define a relative index between two critical points of CS. This index will only be $\mathbb{Z}_8$-valued however, due to looping gradient flow lines. To circumvent the issues arising from reducible connections, we will restrict ourselves to homology three-spheres, for which achieving transversality is easier than for other classes of three-manifolds. A feature of CS which makes it particularly suited for Morse homology is that its critical points and gradient trajectories have geometrical interpretations. The critical points of CS are flat connections on N, and the flow lines are **instantons** over the Riemannian tube $\mathbb{R}\times N$ which join two connections. As we will see, instantons are a special case of **Yang-Mills connections**, which allows us to use Uhlenbeck's compactness theorem to derive the compactness up to broken trajectories of the trajectory spaces. The invariant we will obtain in the end takes the form of a $\mathbb{Z}_8$-graded vector space over the field $\mathbb{Z}_2$. For the usual three-sphere S^3 and the Poincaré homology sphere P they are given by:

$$\boxed{\mathrm{HF}^\bullet(S^3) = 0, \quad \mathrm{HF}^\bullet(P) = (0,\mathbb{Z}_2,0,0,0,\mathbb{Z}_2,0,0).}$$

Thus the Floer homology groups will be unrelated to the usual homology groups, and provide a stronger invariant. To conclude, we will investigate further properties of the Floer groups, such as their relation to the representation-theoretic **Casson invariant**, which appears as the Euler characteristic of the Floer groups, and $(3+1)$-dimensional **topological quantum field theories**.

We will move according to the following outline. In the first chapter, we introduce electromagnetism as an example of an abelian gauge theory, familiarise ourselves with the concepts we will later encounter in the non-abelian setting and provide a link between gauge theory and topology of three-manifolds via Hodge theory. In the second chapter, we will then recall principal bundles properly, fix the notation, and derive the relation between three- and four-dimensional $\mathrm{SU}(2)$-gauge theory. Next, we will introduce the Chern-Simons and Yang-Mills functionals CS and YM in the third chapter and analyse the relation between CS-gradient flow lines and instantons, which are the local minimizers of YM. We will also investigate the local picture around a critical point of these functionals. In the last chapter we will construct the trajectory spaces and the Floer homology groups from a chain complex generated by flat connections, emphasising the Fredholm analysis. We will show independence of auxiliary data and of the perturbation chosen, where it will be important to restrict to the case of homology three-spheres.

We assume that the reader is familiar with differential geometry (bundles and connections, Riemannian geometry, de Rham cohomology, Morse homology), algebraic topology (especially computational tools in (co)homology theories, Poincaré duality, intersection theory, some homotopy theory), as well as functional analysis (Sobolev spaces on open domains, differential operators on $\mathbb{R}^n$).

These notes do not contain original work, but merely adapt and combine the literature on the subject, while supplementing explanations where deemed useful. We mostly follow Donaldson's monograph on the topic [1], and apply the treatment of Morgan's lecture notes [5] to give the analytical foundations for the Chern-Simons and Yang-Mills functionals. The first chapter on electromagnetism reshuffles the first part of the excellent lecture notes by Evans on the Yang-Mills equations [2], and the appendix bundles statements from multiple sources, most notably the development of the spectral flow from [17].

I would like to thank my supervisor Prof. Will Merry for guiding me through this endeavour with frequent and useful meetings. His (virtual) office door was always open to me, and I appreciated both his technical knowledge in geometry and analysis and his expertise in the wider world of Floer theories, which often provided helpful analogies an additional angles of attack during my study of the instanton flavoured theory.

Contents

Chapter 1

Electromagnetism and Topology

In this chapter we will introduce electromagnetism as a motivational example of an abelian gauge theory, meaning a theory whose structure group (in our case $\mathrm{U}(1)$) is abelian. We will encounter some defining characteristics of the theory, such as the existence of local potentials and gauge symmetry. Furthermore we study the Maxwell equations, a linear system of equations, to pave the way for the non-linear Yang-Mills equations that we will study in this thesis. We will see that already the abelian theory which governs the electro-magnetic fields can recover some homological information about the underlying space via Hodge theory.

1.1 The Maxwell equations

The electric field E and magnetic field B can be interpreted as two vector fields on $\mathbb{R}^3$ which vary with time, i.e. $E(t), B(t) \in \mathscr{X}(\mathbb{R}^3)$. The **Maxwell equations** describe their evolution by four linear partial differential equations:

$$\boxed{\begin{array}{rclrcl} \nabla \cdot E & = & \rho & \nabla \times E + \frac{\partial}{\partial t} B & = & 0 \\ \nabla \cdot B & = & 0 & \nabla \times B - \frac{\partial}{\partial t} E & = & j \end{array}} \tag{1.1}$$

Here $\rho(t) \in C^\infty(\mathbb{R}^3)$ is the charge density, and $j(t) \in \mathscr{X}(\mathbb{R}^3)$the current density, which we consider as fixed parameters. Consider for instance the case $\rho = 0$ and $j = 0$, i.e. the situation free of charges and currents. In this case one has to solve the homogeneous **vacuum Maxwell equations**:

$$\boxed{\begin{array}{rcccl} \nabla \cdot E & = & \nabla \cdot B & = & 0 \\ \nabla \times E + \frac{\partial}{\partial t} B & = & \nabla \times B - \frac{\partial}{\partial t} E & = & 0 \end{array}} \tag{1.2}$$

The Maxwell equations can be written in vector form as above because space time $\mathbb{R}^4 = \mathbb{R} \times \mathbb{R}^3$ admits global coordinates. But what about other space-time manifolds? The correct notion to use here is the **Lorentzian manifold**, i.e. a four-manifold M with a non-degenerate symmetric bilinear form g with signature $(1,3)$.

As an example, consider **Minkowski space** $\mathbb{R}^4$ with the standard coordinates (t, x, y, z) and with the metric:

$$g^2 = dt^2 - dx^2 - dy^2 - dz^2.$$

On these more general spaces, we can write the Maxwell equations without reference to local coordinates in terms of an abstract two-form $F \in \Omega^2(M)$, called the **electromagnetic tensor**. To see this, define F on Minkowski space as folows:

$$F = E_x dt \wedge dx + E_y dt \wedge dy + E_z dt \wedge dz + B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy.$$

If we denote the coordinate vector fields on $\mathbb{R}^4$ by ∂_μ, we can equivalently think of F as an antisymmetric matrix with entries $F_{\mu\nu} = F(\partial_\mu, \partial_\nu)$, i.e.:

$$F = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & B_z & -B_y \\ -E_y & -B_z & 0 & B_x \\ -E_z & B_y & B_x & 0 \end{pmatrix}$$

Furthermore, denote by $J = \rho dt + j_x dx + j_y dy + j_z dz \in \Omega^1(\mathbb{R}^4)$ the so called **4-current**. How can we formulate the Maxwell equations using F and J alone, so that on $\mathbb{R}^4$ they reduce to the vector form? For this we need to recall a few differential-geometric operators defined for a semi-Riemannian manifold (M, g). First, recall that the **Hodge-star operator** $\star_g = \star : \Omega^k(M) \to \Omega^{n-k}(M)$ is characterised by the property that in local coordinates x^i and for a k-form $\eta \in \Omega^k(M)$ the form $\star\eta$ satisfies:

$$\eta \wedge \star\eta = \sqrt{|\det(g_{ij})|} dx^1 \wedge \cdots \wedge dx^n.$$

Intuitively, the Hodge-star extends the correspondence between k-frames and $(n-k)$-frames in $\mathbb{R}^n$ via the orthogonal complement to families of such frames, and thus also to differential forms. For instance in Minkowski space we have $\star dt = dx \wedge dy \wedge dz$ and $\star(dx \wedge dy) = dt \wedge dz$. Next, the **de Rham differential**:

$$d : \Omega^\bullet(M) \to \Omega^{\bullet+1}(M)$$

is the unique extension of the differential of functions $d : C^\infty(M) \mapsto \Omega^1(M)$ to higher forms such that the graded Leibniz rule:

$$d(\eta \wedge \tau) = (d\eta) \wedge \tau + (-1)^{|\eta|} \eta \wedge (d\tau)$$

as well as the coboundary identity $d^2 = 0$ are satisfied. Finally, $\delta = (-1)^{(n-k)k+1+s} \star d \star : \Omega^\bullet(M) \to \Omega^{\bullet-1}(M)$ is the adjoint to d with respect to the L^2-metric on k-forms given by decomposable forms as:

$$\langle\langle \bigwedge_{i \in I} dx_i, \bigwedge_{j \in J} dx_j \rangle\rangle := \int_{(M,g)} \det(\langle \partial_i, \partial_j \rangle_g) = \int_M \bigwedge_{i \in I} dx_i \wedge \star \bigwedge_{j \in J} dx_j.$$

In other words δ is characterized by the property that $\langle\langle d\eta, \tau \rangle\rangle = \langle\langle \eta, \delta\tau \rangle\rangle$. Here s denotes the signature of the metric. So for instance on 2-forms in Minkowski space we obtain $\delta = \star d \star$. This leads us to a new, more invariant formulation of the Maxwell equations.

Theorem 1.1 *The Maxwell equations are equivalent to the set of equations for $F \in \Omega^2(\mathbb{R}^4), J \in \Omega^1(\mathbb{R}^4)$:*

$$\boxed{\begin{array}{rcl} dF & = & 0 \\ \delta F & = & J \end{array}} \tag{1.3}$$

Note that in the absence of currents,we can rewrite this pair of equations equivalently as a single equation:

$$\Delta F = 0$$

where $\Delta : \Omega^\bullet(M) \to \Omega^\bullet(M)$ is the operator $\Delta = d\delta + \delta d$, called the **Hodge Laplacian** (cf. A.7) and is intricately linked to the topology of the base manifold. Forms $\tau \in \Omega^k(M)$ which satisfy $\Delta\tau = 0$ are called **harmonic** in analogy to harmonic functions, to which this definition reduces on 0-forms (i.e. functions).

Proof Consider the first equation $dF = 0$. We compute:

$$\begin{aligned} dF =& \left(\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + \frac{\partial B_z}{\partial z}\right) dx \wedge dy \wedge dz \\ &+ \left(\frac{\partial}{\partial y}E_x - \frac{\partial}{\partial x}E_y + \frac{\partial}{\partial t}B_z\right) dt \wedge dx \wedge dy + (\dots) \\ =& (\nabla \cdot B) dx \wedge dy \wedge dz + \left(\nabla \times E + \frac{\partial B}{\partial t}\right)_z dt \wedge dx \wedge dy + (\dots). \end{aligned}$$

From this it is clear that $dF = 0$ exactly when the second and third Maxwell equations hold. We compute furthermore:

$$\begin{aligned} \delta F = \star d \star F =& \left(\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}\right) dt + \left(\frac{\partial}{\partial y}B_x - \frac{\partial}{\partial x}B_y - \frac{\partial}{\partial t}E_z\right) dz + (\dots) \\ =& (\nabla \cdot E) dt + \left(\nabla \times B + \frac{\partial E}{\partial t}\right)_z dz + (\dots). \end{aligned}$$

So the equation $\delta F = J$ is exactly a reformulation of the first and fourth Maxwell equation. □

Now it is easy to generalize the Maxwell equations to any Lorentzian manifold (M, g). A solution to the Maxwell equations on M with given current $J \in \Omega^1(M)$ is precisely a two-form $F \in \Omega^2(M)$ such that the invariant formulation of the equations hold:

$$\boxed{\begin{array}{rcl} dF & = & 0 \\ \delta F & = & J \end{array}} \tag{1.4}$$

1.1.1 Local potentials

An important feature of these equations is that we can construct solutions from certain auxiliary and non-physical data. Suppose for a moment that $\mathrm{H}^2(M) = 0$ (or work with a subset of M that has vanishing second homology), then since any solution F to the Maxwell equations (1.4) is a closed 2-form, it must in fact be exact, i.e. there is some $A \in \Omega^1(M)$ such that $dA = F$. Reformulated in these terms this **potential** A for F, the first equation becomes redundant (since $d^2 = 0$), so only need to look for solutions to:

$$\delta dA = J \Leftrightarrow \Delta A = J$$

Now one can apply all the tools developed to study the Poisson equation and use them to solve the system of two equations (1.4). In the general case where M has non-trivial homology in degree two, one can still apply the same reasoning to a cover of M via charts with vanishing H^2. One just needs to be careful to match up the solutions over the intersections. We will see in section 2.4 that **connection forms** provide an elegant generalisation to this more complicated case, and in this setting **curvature forms** will be the analogue of F.

These potentials are in fact not physical quantities, but simply computational aids. This stems from the fact that a great many potentials correspond to a single physical situation (i.e. a fixed F). For instance if $dA = F$, then for any smooth function $f \in C^\infty(M)$ the potential $A + df$ is a different plausible potential. We say that the two potentials A and $A + df$ are related by a **gauge transformation** or **gauge equivalent**. From the point of view of de Rham cohomology none of these potentials is privileged, putting them on equal footing. We will meet this **gauge symmetry** again in a more general form when we introduce the theory of principal bundles.

1.2 Lagrangian formalism

An important viewpoint on electromagnetism and partial differential equations in general is the **variational approach**, in which solutions to a PDE are seen as critical points of an energy functional. Let us introduce the electro-magnetic energy functional corresponding to an empty space time M with $J = 0$ as an example:

Theorem 1.2 (Lagrangian formulation) *Suppose that $\mathrm{H}^2(M) = 0$. Then the solutions to the vacuum Maxwell equations are exactly the critical points of the **electromagnetic action functional**:*

$$\mathscr{S}_{em} : \Omega^1(M) \to \mathbb{R}, \quad A \mapsto \frac{1}{2} \int_M dA \wedge \star dA$$

Proof We compute the derivative of $\mathscr{S}_{em}$ at A in direction $\xi \in \Omega^1(M)$, where we write $F = dA$:

$$\begin{aligned}
&\frac{d}{dt}\mathscr{S}_{em}(A + t\xi) = \frac{1}{2}\frac{d}{dt}\int_M (F + td\xi) \wedge \star(F + td\xi) \\
&= \frac{1}{2}\frac{d}{dt}\left(\int_M F \wedge \star F + t\int_M (d\xi \wedge \star F + F \wedge \star d\xi) + t^2\int_M d\xi \wedge \star d\xi\right) \\
&= \frac{1}{2}\int_M (d\xi \wedge \star F + F \wedge \star d\xi) = \int_M \star F \wedge\ d\xi \\
&= -\int_M (d \star F) \wedge\ \xi
\end{aligned}$$

If now A is critical, then we see from the above computation that $d \star F = 0 \Rightarrow \delta F = 0$, hence the vacuum Maxwell equations are satisfied, since $dF = d^2 A = 0$. Similarly, if F is a solution to the Maxwell equations, then it is clearly also critical for the action, as then $d \star F = 0$. □

The variational approach can be used to prove existence results, as it gives the PDE more context. For instance coercivity assumptions on the functional (meaning it blows up as one moves out to infinity) can under favourable conditions lead to global minimizers, which are critical. A more advanced approach is to generalize Morse theory to an infinite-dimensional setting, using the topology of sub-level sets of the functional to deduce the existence of critical points. We will not go into this any further though, as we will develop our theory from the other end, meaning that we will use the number and kinds of solutions to the Yang-Mills PDE as a topological invariant of the manifold they are defined on. The interested reader is referred to [3] for more details on the variational approach to PDEs. However for now, let us take a look at what topological data can be extracted from electro-magnetism already.

1.3 Magnetostatics on three-manifolds

In order to return to a Euclidean situation, we restrict ourselves to magnetostatics, i.e. the case without an electric field and a constant magnetic field. The Maxwell equations then read:

$$\boxed{\begin{array}{rcl} \nabla \cdot B &=& 0 \\ \nabla \times B &=& j \end{array}} \tag{1.5}$$

Here $B \in \mathscr{X}(N)$ is a time-independent vector field on the three-manifold N, which we assume to be closed. Expressed in terms of the differential forms:

$$\begin{aligned}
&\beta = B_x dy \wedge dz + B_y dz \wedge dx + B_z dx \wedge dy \in \Omega^2(N) \\
&j = j_x dx + j_z dy + j_z dz \in \Omega^1(N),
\end{aligned}$$

we have equivalently:

$$\boxed{\begin{array}{rcl} d\beta &=& 0 \\ \delta\beta &=& j \end{array}} \tag{1.6}$$

Here $\star_3$ denotes the Hodge operator of N and $\delta = -\star_3 d\star_3$ the adjoint to d with respect to $\star_3$. We will now further specialize to the situation $j = 0$, i.e. the situation without currents to simplify matters further. The main result of this section which provides the link between magnetostatics on a three-manifold and its topology is the following result:

Theorem 1.3 (Hodge) *In each de Rham coholomogy class there is a unique closed 2-form β such that equation 1.6 holds for $j = 0$.*

Proof We are interested in determining the solutions to the equations of magnetostatics in a vacuum:

$$d\beta = 0 \wedge \delta\beta = 0 \Leftrightarrow \Delta\beta = 0.$$

It is an important theorem of **Hodge theory** that every cohomology class is represented by a unique harmonic form (see A.47). The heuristic motivation for this is that the homology class of β is represented in $\Omega^2(M)$ by an affine plane of the form $\beta + d\alpha$ for $\alpha \in \Omega^1(M)$, and being harmonic is equivalent to being a global minimizer of the L^2-norm functional on that affine plane:

$$\beta \mapsto \int_M \beta \wedge \star_3\beta = \|\beta\|_{L^2}^2.$$

Notice the similarity to the electro-magnetic energy functional from theorem 1.2. In order prove existence of such a minimizer, proper function spaces of weak solutions have to be set up and coercivity of the functional is then shown (essentially the graph of this functional looks like a potential pot). After that uniqueness follows from the strict convexity of the functional in this Euclidean setting. After that elliptic regularity theory can be applied to show that this unique weak solution is in fact a strong solution, i.e. a smooth harmonic form. □

Denote the space of solutions to the static magnetic Maxwell equations, considered up to gauge equivalence, called the **moduli space** of solutions, by $\mathscr{M}$. By this we mean the set of potentials, but where A and $A + df$ are identified for any $f \in C^\infty(N)$. We have just proven that:

$$\mathscr{M} \simeq \mathrm{H}^2(N, \mathbb{R}),$$

so magnetostatics indeed recovers topological information.

Chapter 2

Gauge Theory

In this chapter we recall the theory of principal bundles and connections on them in order to extend the results from the last chapter to non-abelian symmetry groups. We will also prove some auxiliary results relating connections on a Riemannian tube $\mathbb{R} \times N$ to paths of connections on the cross-section N. With this new tools at our disposal, we will review the electromagnetism as the gauge theory of a trivial $\mathrm{U}(1)$-principal bundle. Finally, we will gather group-theoretic and differential-geometric facts about the Lie group $\mathrm{SU}(2)$, the structure group of most of the principal bundles appearing in the coming chapters.

2.1 Principal bundles

Let M be a smooth manifold and let G be a Lie group. A **principal G-bundle** over M is a fibre bundle $P \to M$ with typical fibre G together with a smooth and free right-action $P \curvearrowleft G$ which preserves the fibres and acts transitively on a fibre. A principal bundle is thus locally trivialisable as $P|_U \simeq U \times G$, and it can be shown that this trivialisation can be chosen such that the action on P corresponds to the right-action of G on itself. In analogy to vector bundles, principal bundles can be described using an atlas and transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G \subset \mathrm{Diff}(G)$, see B.11 for more details. In physics, this local product structure is interpreted as attaching additional local data (with values in G) to points in space-time (the base manifold M). In the example of electrodynamics, the correct setting is a $\mathrm{U}(1)$-bundle, so the additional data is a complex number of unit length. Note that while fibre-wise $G \simeq P_x$, there is no canonical way of doing so, i.e. there is no distinguished identity element in a fibre of P. This ambiguity is what will lead to **gauge transformations**.

Associated bundles

Principal bundles are non-linear objects, hence one might hope to define linearisations which recover some of their features. Suppose that we have an atlas of a principal bundle P with its transition functions $g_{\alpha\beta} : U_\alpha \cap U_\beta \to G$ and let $\rho : G \to \mathrm{GL}(V)$

be a representation of G on a finite dimensional vector space. By post-composition, we obtain a new set of transition functions $\rho \circ g_{\alpha\beta} : U_\alpha \cap U_\beta \to \mathrm{GL}(V)$, which determine a vector bundle with typical fibre V. It furthermore inherits a right action from the principal bundle we started from, which motivates its name: the **associated bundle** to P via the representation (V, ρ), denoted $P \times_\rho V$.

Example 2.1 *Denote by* $\mathrm{ad} : G \to \mathrm{GL}(\mathfrak{g})$ *the adjoint representation of G on its Lie algebra. The associated bundle to P via* ad *is called the* ***adjoint bundle***

$$\mathrm{ad}\, P := P \times_{\mathrm{ad}} \mathfrak{g}.$$

It can be seen as the bundle of Lie algebras of the fibres with the corresponding adjoint action. Since the commutator on $\mathfrak{g}$ *is* ad*-equivariant in the sense that:*

$$[\mathrm{ad}_g v, \mathrm{ad}_g w] = \mathrm{ad}_g [v, w],$$

the bundle $\mathrm{ad}\, P$ *admits a fibre-wise product* $[\cdot, \cdot]$*, which makes its space of sections* $\Gamma(\mathrm{ad}\, P)$ *into an infinite dimensional Lie algebra.*

Example 2.2 *Let G be a matrix Lie group, i.e. there is an inclusion* $\iota : G \hookrightarrow \mathrm{GL}(n, \mathbb{K})$*, where* $\mathbb{K} = \mathbb{R}$ *or* $\mathbb{C}$*. Then we can form the bundle* $P \times_\iota \mathbb{K}^n$*, which simply means that we re-interpret the transition functions* $g_{\alpha\beta} : U_{\alpha\beta} \to G$ *as maps into* $\mathrm{GL}(n, \mathbb{K})$ *instead. In the case of* $G = \mathrm{O}(n)$ *or* $G = \mathrm{U}(n)$ *this associated bundles is called the bundle of* ***orthogonal/unitary frames****, and no information is lost by considering the frame bundle instead of the initial principal bundle.*

Alternatively an associated bundle $P \times_\rho V$ can be seen as the quotient of $P \times V$ w.r.t. the equivalence relation $[p \cdot g, v] \sim [p, \rho(g)v]$ (see also [4], lecture 25). We can consider $\mathrm{ad}\, P$- and $\mathfrak{g}$-valued forms. They admit some interesting further structure.

Example 2.3 *Consider a principal bundle* $P \xrightarrow{\pi} M$ *and its adjoint bundle* $\mathrm{ad}\, P$*. The latter has a natural product* $[\cdot, \cdot]$ *induced from the commutator on* $\mathfrak{g}$ *as seen above. It can be extended to a product*

$$\Omega^k(M, \mathrm{ad}\, P) \times \Omega^l(M, \mathrm{ad}\, P) \to \Omega^{k+l}(M, \mathrm{ad}\, P)$$

by letting it act trivially on forms. This extended product is graded-anticommutative, meaning it satisfies:

$$[\omega, \eta] = (-1)^{kl+1} [\eta, \omega]$$

The similarly defined commutator on $\Omega^\bullet(M, \mathfrak{g})$ *is graded-anticommutative.*

Example 2.4 *Let G be a compact Lie group. Then its Lie algebra* $\mathfrak{g}$ *admits an* ad*-invariant inner product. This can be used to define a Euclidean metric on* $\mathrm{ad}\, P$*, denoted* $\langle \cdot, \cdot \rangle_{\mathrm{ad}\, P}$*. For instance in the case of* $\mathrm{SU}(n)$ *this product is given as:*

$$\langle s, t \rangle_{\mathrm{ad}\, P} = -\operatorname{tr}(st).$$

This inner product can be extended to a graded-commutative product

$$-\operatorname{tr}(\cdot \wedge \cdot) : \Omega^k(M, \operatorname{ad} P) \otimes \Omega^l(M, \operatorname{ad} P) \to \Omega^{k+l}(M).$$

Using the ***bundle-valued Hodge star operator*** $\star : \Omega^\bullet(M, \operatorname{ad} P) \to \Omega^{n-\bullet}(M, \operatorname{ad} P)$ *(which acts trivially on the sections of* $\operatorname{ad} P$*) we can furthermore define a product on* $\Omega^k(M, \operatorname{ad} P)$*, given on decomposable elements as:*

$$\langle \sigma \otimes s, \tau \otimes t \rangle_{\operatorname{ad} P} = -\operatorname{tr}\left((\sigma \otimes s) \wedge \star (\tau \otimes t)\right) = (\sigma \wedge \star\tau) \langle s, t \rangle_{\operatorname{ad} P}.$$

By integration we then obtain an inner product on $\Omega^k(M, \operatorname{ad} P)$*:*

$$\langle\!\langle \alpha, \beta \rangle\!\rangle_{\operatorname{ad} P} := \int_M \langle \alpha, \beta \rangle.$$

Remark 2.5 *Note that if* $G \subset \operatorname{GL}(n)$*, then it is possible to define a product:*

$$\wedge : \Omega^k(M, \mathfrak{g}) \otimes \Omega^l(M, \mathfrak{g}) \to \Omega^{k+l}(M, \mathfrak{gl}(n)),$$

by extending the product of matrices. This notation is consistent with the product $-\operatorname{tr}(\cdot \wedge \cdot)$ *we introduced earlier! Notice however that in general the result of this operation will only be a* $\mathfrak{gl}(n)$*-valued form. Nonetheless there are situations where the result is a* $\mathfrak{g}$*-valued form. For instance if* $\varpi \in \mathscr{C}(P)$*, then* $\varpi \wedge \varpi \in \Omega^2(P, \mathfrak{g})$*, since:*

$$\frac{1}{2}[\varpi, \varpi] = \varpi \wedge \varpi.$$

In the following, whenever such a wedge product appears, it should be understood that either the result is again a $\mathfrak{g}$*-form, or that the result is a* $\mathfrak{gl}(n)$*-form, but that it does not matter in context (as for instance a trace is immediately taken).*

Differential forms with values in $\operatorname{ad} P$ are related to differential forms on P with values in $\mathfrak{g}$ that satisfy certain symmetry conditions, which we will now explain. Consider a differential form $\sigma \in \Omega^k(P, \mathfrak{g})$. We say that σ is **horizontal** if it vanishes on vertical vectors, i.e. if $v_1, \ldots, v_k \in T_pP$ and $d\pi_p[v_i] = 0$ for some $1 \leq i \leq k$, then $\sigma_p(v_1, \ldots, v_k) = 0$. Furthermore, σ is called **equivariant** if $r_g^*\sigma = \operatorname{ad}_{g^{-1}} \sigma$. The set of differential forms which are both equivariant and horizontal is denoted by $\Omega^k_G(P, \mathfrak{g})$. They are in fact isomorphic as vector spaces to sections of the adjoint bundle:

$$\Omega^k_G(P, \mathfrak{g}) \simeq \Omega^k(M, \operatorname{ad} P)$$

We fix the notation that the isomorphisms will be referred to as $\flat : \Omega^k_G(P, \mathfrak{g}) \to \Omega^k(M, \operatorname{ad} P)$ and $\sharp : \Omega^k(M, \operatorname{ad} P) \to \Omega^k_G(P, \mathfrak{g})$.

Princial bundles and trivialisations

In the following chapters we will mostly work with trivial principal bundles, which are exactly the bundles that admit a global section.

Proposition 2.6 (Local trivialization and local sections) *Let P be a principal G-bundles over M and let $U \subset M$ be an open set. Then the following are equivalent:*

- *There is a local section $\sigma : U \to P|_U$.*
- *The bundle $P|_U$ is trivial.*

Proof Let $\sigma \in \Gamma_U(P)$. Then by freeness and smoothness of the action the map $\eta(x,g) = \sigma(x) \cdot g : U \times G \to P|_U$ is a local trivialization. If on the other hand $\psi : U \times G \to P|_U$ is a trivialization, then $\psi(\cdot, 1) : U \to P|_U$ is a smooth local section. □

Let $\psi : U \times G \to P|_U$ be a local trivialization. It induces local trivializations on every associated bundle $B = P \times_\rho V$ via:

$$\begin{aligned} \psi_B : U \times V &\to B|_U \\ (x,v) &\mapsto [\sigma(x), v] \end{aligned}$$

It is indeed a trivialization, since it is an injective vector bundle homomorphism between vector bundles of equal dimension.

2.2 Connections

Recall that a **connection one-form** on a principal bundle $P \to M$ is an element $\varpi \in \Omega^1(P, \mathfrak{g})$ which satisfies two conditions:

1. **Equivariance**: For $g \in G$: $r_g^* \varpi = \mathrm{ad}_{g^{-1}} \varpi$
2. **Verticality**: For $v \in \mathfrak{g}$: $\varpi_p(\xi_v(p)) = v$, where $\xi_v \in \mathscr{X}(P)$ is the fundamental vector field corresponding to the infinitesimal action of $v \in \mathfrak{g}$.

The physical interpretation of a connection is that when a particle moves in the space-time manifold, there should be an associated movement of the additional data above the curve, i.e. a parallel transport.

We denote by $\mathscr{C}(P)$ the space of all smooth connection one-forms on the principal bundle P. We also sometimes write $\mathscr{C}(M)$ to denote the connections on a trivial $G-$bundle over M, and this convention can be applied whenever we introduce a space dependent on an principal bundle. By fixing a reference connection ϖ_0, one can check that

$$\mathscr{C}(P) = \varpi_0 + \Omega^1(M, \mathrm{ad}\, P)$$

has the structure of an affine space. Just note that the difference of two connection forms is both equivariant and horizontal (because of the normalization condition), hence is represented by a unique element of $\Omega^1(M, \mathrm{ad}\, P)$. One can thus identify the tangent space to a connection $\varpi \in \mathscr{C}(P)$ with the underlying topological vector space, i.e. $T_\varpi \mathscr{C}(P) \simeq \Omega^1(M, \mathrm{ad}\, P)$. In particular, $\mathscr{C}(P)$ can be endowed with a Fréchet space structure, which is uniquely defined if M is compact (see also appendix

A.1). We will denote connection forms on four-manifolds by ϖ and connections on three-manifolds by ω. As principal bundles are determined by an atlas, connections forms can also be given locally via a collection of one-forms on the base manifold. Their precise transformation behaviour is given by the following result.

Proposition 2.7 *Let $P \xrightarrow{\pi} M$ be a principal G-bundle (where $G \subset \mathrm{GL}(n, \mathbb{K})$) with trivializing atlas $\{U_\alpha, \psi_\alpha\}$ and transition functions $g_{\alpha\beta}$. Let $\varpi \in \mathscr{C}(P)$ be a smooth connection. Let $\sigma_\alpha : U_\alpha \to U_\alpha \times G$ be the constant 1-section. Define $\varpi_\alpha := \sigma_\alpha^* \psi_\alpha^* \varpi$. Then on $U_\alpha \cap U_\beta$:*

$$\varpi_\beta = \mathrm{ad}_{g_{\alpha\beta}} \varpi_\alpha - dg_{\alpha\beta} g_{\alpha\beta}^{-1}.$$

Here $-dg_{\alpha\beta} g_{\alpha\beta}^{-1} = g_{\alpha\beta}^* \varphi_{MC}$ is the pull-back of the ***Maurer-Cartan form*** *$\varphi_{MC} \in \Omega^1(G, \mathfrak{g})$, which is the only left-invariant one-form on G such that $(\varphi_{MC})_1 = \mathrm{id}$. Conversely, any collection of one-forms on M satisfying the above identity give rise to a unique smooth connection on P.*

Remark 2.8 *Note that if P is a trivial bundle, hence admits a global section $\sigma \in \Gamma(P)$, a connection is equivalently given by a single Lie-algebra valued one-form on M, which will be denoted by A on three-manifolds and by $\mathbf{A}$ on four-manifolds. We will call these forms* ***connection potentials****. In particular we have $\mathscr{C}(P) \simeq \Omega^1(M, \mathfrak{g})$. However note also that the representation as a $\mathfrak{g}$-valued form depends on the trivialization.*

Next, we recall that a connection on a principal bundle induces connections on associated bundles, in particular on the adjoint bundle $\mathrm{ad}\, P$. Let $\varpi \in \mathscr{C}(P)$ be a connection. The induced connection on $\mathrm{ad}\, P$ is then the unique connection for which sections of $\mathrm{ad}\, P$ of the form $[\gamma(t), v]$ are parallel exactly when $\gamma(t)$ is a parallel section of P. We denote the covariant derivative corresponding to this connection by $\nabla^\varpi : \Omega^0(M, \mathrm{ad}\, P) \to \Omega^1(M, \mathrm{ad}\, P)$. Similar to how the de Rham differential is the unique extension of the Lie derivative of functions $\mathscr{L} : \Omega^0(M) \to \Omega^1(M)$ satisfying the graded Leibniz rule, there is an extension of ∇^ϖ to a graded derivation $d_\varpi : \Omega^\bullet(M, \mathrm{ad}\, P) \to \Omega^{\bullet+1}(M, \mathrm{ad}\, P)$, called the **exterior covariant derivative**. Indeed if we interpret the Lie derivative as a trivial connection on a trivial line-bundle (whose sections are precisely the smooth functions) the de Rham differential appears as its exterior covariant derivative. Explicitly it can be given as:

$$d_\varpi \sigma = \left(d\sigma^\sharp + \varpi \wedge_{\mathrm{ad}} \sigma^\sharp \right)^\flat \tag{2.1}$$

Here $\wedge_{\mathrm{ad}}$ should be interpreted as acting on sections via the adjoint representation of $\mathfrak{g}$ on itself, i.e.

$$f \wedge_{\mathrm{ad}} s := \mathrm{ad}_f(s),$$

where $f \in \Gamma(P, \mathfrak{g})$ and $s \in \Gamma(P, \mathfrak{g})$. The extension to forms is then obtained by acting trivially on the form part. If the base manifold is Riemannian, then we furthermore define the the $\langle\!\langle \cdot, \cdot \rangle\!\rangle_{\mathrm{ad}\, P}$-adjoint of d_ϖ, via $\delta_\varpi = (-1)^{n(n-k)} \star d_\varpi \star$. The inner product as well as the adjoint depend on the Riemmanian metric. Note that while both the de Rham differential d and its adjoint δ square to zero, the same is not true for the exterior derivative. Here $d_{\varpi}^2\tau = R^{\nabla} \wedge \tau$ and $\delta_{\varpi}^2\tau = (-1)^{k(n-k)} \star R^{\nabla} \wedge \star\tau$.

Remark 2.9 *In this section we were concerned solely with smooth connections. However for the analytic treatment which follows it is favourable to work with a Banach manifold structure on $\mathscr{C}(P)$, which can be achieved by considering Sobolev class sections. More details can be found in the appendix A.1.*

Families of connections

Let N be a closed manifold and Q a trivial principal G-bundle over N. Let T be a smooth manifold. We say that a family of connections ω_t on Q, parametrized by $t \in T$ is a **smooth family of connections**, if the map $T \to \mathscr{C}(Q)$ induced this way is smooth as a map between Fréchet manifolds. Fix trivializations of Q and π^*Q, where $\pi : T \times N \to N$ is the projection.

Proposition 2.10 *Let T be a smooth finite dimensional manifold and let P be the trivial G bundle over $M = N \times T$. A smooth family of connections $\omega_t \in \mathscr{C}(Q)$ determines a connection $\varpi \in \mathscr{C}(P)$. This assignment is injective.*

Proof Let the potential corresponding to ω_t be A_t. We can reinterpret this collection as a single one-form $\mathbf{A} \in \Omega^1(N \times T, \mathfrak{g})$ via

$$\mathbf{A}(x,t)(v,\xi) := A_t(x)[v].$$

This form is smooth, as A_t was assumed to be a smooth family (cf. A.3). Now again by 2.7 this determines a connection $\varpi \in \mathscr{C}(P)$.

Denote by $\iota_t : N \to N \times T$ the inclusion at $t \in T$. Then a connection $\varpi \in \mathscr{C}(P)$ determines connections on slices via:

$$\omega_t = \iota_t^* \varpi.$$

For the local connection potential this implies $A_t = \iota_t^* \mathbf{A}$, which amounts to saying:

$$A_t(x)[v] = \mathbf{A}(x,t)[v,0],$$

from which we see that A_t is in fact a smooth family (as $\mathbf{A}_t$ is a smooth differential form). Starting with a family of connections ω_t, gluing them together to a single connection on P, and splitting them up again into a family of connections, we obtain the same connection we started with, hence the gluing construction is injective. □

The assignment $A_t \mapsto \mathbf{A}$ is in general only injective, and not surjective. This is due to the fact that only connections which are trivial on the T-factor or **in temporal gauge** with respect to the trivialisation (meaning that $\mathbf{A}(x,t)(0,\xi) = 0$ for every tangent vector to T) can be produced by T-parametrized families of connections through this construction. However since in dimensions two and above every manifold admits a non-flat connection, surjectivity is not given. In contrast to that, every connection on a one-dimensional manifold is flat, since in this case the curvature must vanish. Thus there always is a way to see a connection with $T = \mathbb{R}$ as being in temporal gauge by choosing an appropriate trivialisation.

Proposition 2.11 *For a connection $\varpi \in \mathscr{C}(P)$ and $T = \mathbb{R}$, there is always a trivialization such that the local connection potential is in temporal gauge.*

Proof Let $\gamma_x : \mathbb{R} \to N \times \mathbb{R}$ be the path $\gamma_x(t) = (x, t)$. Let $\tilde{\sigma} : N \times \{0\} \to P$ be a smooth section (which exists, since $P|_{N\times\{0\}}$ is trivial). We extend it to a section on all of M by:

$$\sigma(x,t) = \mathbb{P}_{\gamma_x}\left(\tilde{\sigma}(x,0)\right)(t).$$

By the properties of parallel transport, this section is smooth. Furthermore consider:

$$\sigma^*\omega(x,t)[0,\partial t] = \omega(\sigma(x,t))[d\sigma(0,\partial t)].$$

But since $d\sigma(0,\partial t)$ is the velocity vector of a parallel lift, it is horizontal, hence $\sigma^*\omega(x,t)[0,\partial t] = 0$. Hence the local connection potential corresponding to σ does not have a dt-term and so is in temporal gauge. □

From this lemma we see that if $T = \mathbb{R}$, no information is lost when splitting up a connection, since every connection can be put in temporal gauge. Note however that the precise form of the correspondence between paths in $\mathscr{C}(Q)$ and connections in $\mathscr{C}(P)$ depends on the choices of trivialisations.

Curvature

Given a connection $\varpi \in \mathscr{C}(P)$, we denote its curvature by $\Omega_\varpi = \Omega \in \Omega^2_G(P, \mathfrak{g})$. By the **Cartan structure formula** it can be computed from the connection form as:

$$\Omega = d\varpi + \frac{1}{2}[\varpi, \varpi].$$

Furthermore, it satisfies the **Bianchi identity** $d\Omega = [\Omega, \sigma]$. The curvature can either be seen as the obstruction to integrability of the distribution $\ker \varpi$ or as infinitesimal parallel transport around a parallelogram. In our treatment we will also consider the corresponding $\operatorname{ad} P$-valued differential form $\Omega^\flat \in \Omega^2(M, \operatorname{ad} P)$, also denoted by F_ϖ. This last notation stems from physics, where curvature is a measurable quantity. One example of this was the the electromagnetic tensor F considered last chapter. The relation will be made clear when we re-examine the Maxwell equations in a bit. A direct computation using the Cartan structure formula and the identity 2.1 leads to the following useful formula:

Proposition 2.12 *Let $\varpi \in \mathscr{C}(P)$ be a connection and let $v \in T_\varpi\mathscr{C}(P) \simeq \Omega^1(M, \operatorname{ad} P)$ be a tangent vector. Then:*

$$\mathrm{F}_{\varpi+tv} = \mathrm{F}_\varpi + t d_\varpi v + t^2 v \wedge v. \tag{2.2}$$

Now let us recall the relation between the different curvature tensors on a principal bundle and its associated bundles.

Proposition 2.13 (Curvature in trivialisation) *Let $\varpi \in \mathscr{C}(P)$ be a connection on a principal bundle P, which admits a local trivialization $\psi : U \times G \to P|_U$. Under the induced trivialization $\psi_{\operatorname{ad} P} : U \times \mathfrak{g} \to \operatorname{ad} P|_U$, the form F_ϖ corresponds to a $\mathfrak{g}$-valued two form $\mathrm{F}_{\mathbf{A}} \in \Omega^2(U, \mathfrak{g})$. Then:*

$$\mathrm{F}_{\mathbf{A}} = d\mathbf{A} + \frac{1}{2}[\mathbf{A}, \mathbf{A}].$$

Here **A** *is the local connection potential with respect to the trivialization.*

Flat connections

A connection $\varpi \in \mathscr{C}(P)$ is called **flat** if $\mathrm{F}_\varpi = 0$. We denote the set of flat connections by $\mathrm{R}(P)$. Their defining properties are that their horizontal distributions $\ker \varpi$ are integrable, meaning that a flat connection is locally trivial. However globally some non-trivial holonomy may arise from non-contractible loops. Think for instance of the natural connection on a Möbius strip over S^1. Since flatness also implies that parallel transport is infinitesimally trivial, it can be shown that the holonomy around a based loop only depends on the homotopy class of that loop. So what we obtain is a representation of the fundamental group into G, i.e. there is a map, called the **holonomy representation**:

$$\mathrm{R}(P) \to \mathrm{Hom}(\pi_1(M), G).$$

Note that it is also possible to define a flat connection from an given element of $\mathrm{Hom}(\pi_1(M), G)$. We will see below that up to the right notions of equivalences on both sides this is in fact a bijective correspondence.

2.3 Gauge transformations

Let $P \to M$ be a trivial principal G-bundle. Then given a trivialisation $\psi : P \to M \times G$, we can define a trivial connection on P via pull-back: $\varpi_\psi \in \mathscr{C}(P)$. We see that in this way many different connections arise. Consider for instance $M = S^1$ and $G = \mathrm{U}(1)$. To specify a trivial connection it is enough specify a single global section which we consider parallel. Thus there is correspondence between trivial connections and elements of $C^\infty(S^1, \mathrm{U}(1))$, which is an infinite dimensional space. Suppose however that we have two such connections ϖ_1, ϖ_2 corresponding to parallel sections $s_1, s_2 \in C^\infty(S^1, \mathrm{U}(1))$. Then there is a morphism of principal bundles $\varphi : P \to P$ mapping s_1 to s_2, which has as a consequence that $\varphi^*(\varpi_2) = \varpi_1$. Thus, up to rotating each fibre by some amount corresponding to $s_2(s_1)^{-1}$, the two connections have the same properties. We say that they are **gauge equivalent** and the morphism φ a **change of gauge** or **gauge transformation**, since a change of the way the principal bundle is measured (the gauge) makes one connection go over into the other. More generally for non-trivial principal bundles we have the following definitions:

Definition 2.14 *Let $P \xrightarrow{\pi} M$ be a G-principal bundle. The group of **gauge transformations** is the group of G-equivariant diffeomorphisms of P (i.e. morphisms of principal bundles) covering the identity:*

$$\mathscr{G}(P) = \left\{\varphi \in \mathrm{Diff}(P) : \varphi(p \cdot g) = \varphi(p) \cdot g, \pi = \pi \circ \varphi\right\}.$$

*Two connections $\varpi_1, \varpi_2 \in \mathscr{C}(P)$ are **gauge equivalent** if there is a gauge transformation $\varphi \in \mathscr{G}(p)$ such that $\varphi^* \varpi_2 = \varpi_1$.*

We now give a more tractable alternative formulation. Let $\mathrm{Ad} : G \to \mathrm{Diff}(G)$ denote the homomorphism given by conjugation, i.e. $\mathrm{Ad}(g)h = ghg^{-1}$. Denote by $\mathrm{Ad}\,P$ the associated fibre bundle $\mathrm{Ad}\,P = P \times_{\mathrm{Ad}} G$. A local section $\psi \in \Gamma_U(\mathrm{Ad}\,P)$ has the shape $\psi(x) = [p(x), u(x)]$, where $p : U \to P$ is a smooth section of the original bundle P and $u : U \to G$ is a smooth map, such that:

$$(p(x) \cdot g, u(x)) = (p(x), \mathrm{Ad}(g)u(x))\,.$$

The utility of this construction is that gauge transformations can be seen as sections of $\mathrm{Ad}\,P$ in the same way an infinitesimal deformation of P can be seen as a section of $\mathrm{ad}\,P$.

Proposition 2.15 ([5], Lemma 4.1.2) *There is an isomorphism of Fréchet manifolds:*

$$\mathscr{G}(P) \simeq \Gamma(\mathrm{Ad}\,P)$$

Remark 2.16 *As we have seen in example 2.1, $\Gamma(\mathrm{ad}\,P)$ admits the structure of an infinite dimensional Lie algebra. It turns out that it is exactly the Lie algebra to the infinite-dimensional Lie group $\Gamma(\mathrm{Ad}\,P)$. See also section 4.2 of [5].*

Note that in the case of a trivial bundle P, every associated bundle must be trivial as well, and hence in such a case the data of a gauge transformation reduces to giving a map $\psi : M \to G$ (like in the introductory example above). Notice that if $[M, G] = \pi_0(\mathscr{G}(P))$ is non-trivial, we can make a distinction between **small** and **big** gauge transformations, a small one being a transformation such that ψ is null-homotopic, i.e. can be joined to the trivial automorphism by a smooth path of gauge transformations. A big one is one which cannot be joined to the identity transformation by a path inside $\mathscr{G}(P)$.

Remark 2.17 *Again as for the space of connections of itself, one can endow $\mathscr{G}(P)$ with a Banach space structure, which is done in the appendix A.2. The exact structure depends on whether one works with three- or four-manifolds, but is always chosen so that the action of $\varphi \in \mathscr{G}(P)$ on $\mathscr{C}(P)$ given by $\varpi \mapsto \varphi^* \varpi$ is a smooth action between Banach manifolds.*

The term gauge transformation stems from physics, where it refers to the change of reference frame by which physical quantities are measured, i.e. a change of coordinates for the fibers of a principal bundle. Let us expand a bit on this interpretation. Consider $P \to M$ trivial, with a fixed parametrisation $\psi : M \times G \to P$. Let $\varpi \in \mathscr{C}(P)$ be a connection with corresponding connection potential $\mathbf{A} = \psi(\cdot, 1)^*\varpi \in \Omega^1(M, \operatorname{ad} P)$. For a gauge transformation $\varphi \in \mathscr{G}(P)$, the connection $\varphi^*\varpi$ has connection potential $\mathbf{A}' = \psi(\cdot, 1)^*\varphi^*\varpi = (\varphi \circ \psi(\cdot, 1))$. However, $\mathbf{A}'$ can also be interpreted as the connection potential corresponding to the connection ϖ, but with respect to the parametrisation $\varphi \circ \psi$. It is thus equivalent to think of gauge transformations as altering the connections or as altering the underlying principal bundle.

A fundamental concept in the physical theory is that since the change of reference should not influence the physics, we can think of gauge equivalent connections as being the same. What is physical (i.e. measurable) however is the curvature, so it should not be affected by gauge transformations, and indeed it is not:

Proposition 2.18 *Let $\varpi \in \mathscr{C}(P)$ be a connection and $\varphi \in \mathscr{G}(P)$ be a gauge transformation. Then:*

$$\mathrm{F}_\varpi = \mathrm{F}_{\varphi^*\varpi}$$

The proof is a direct computation. Inspired by this physical thinking, we would like to work on the **moduli space of gauge equivalence classes of connections**:

$$\mathscr{B}(P) = \mathscr{C}(P)/\mathscr{G}(P)$$

In suitable Sobolev extensions, the action of $\mathscr{G}(P)$ on $\mathscr{C}(P)$ is smooth, thus at points $[\varpi] \in \mathscr{B}(P)$ with discrete stabiliser $\operatorname{stab}_{\mathscr{G}(P)}(\varpi) \subset \mathscr{G}(P)$ the resulting moduli space is a smooth Banach manifold. We call such connections **irreducible**. However in the other case, i.e. when ϖ has non-discrete stabiliser (which we call the **reducible** case) the moduli space admits singularities. Denote the subset of gauge equivalence classes of irreducible connections by $\mathscr{B}^*(P) \subset \mathscr{B}(P)$. Reducibility can alternatively be viewed through the following proposition:

Proposition 2.19 (4.3.3, 4.3.4 in [5]) *Let $\varpi \in \mathscr{C}(P)$ be a connection. Then:*

$$\ker \nabla^\varpi_{\mathrm{Ad}} \simeq \operatorname{stab}_{\mathscr{G}(P)}(\varpi) \simeq \mathrm{Z}\left(\operatorname{hol}(\varpi)\right) \subset G.$$

Here $\nabla^\varpi_{\mathrm{Ad}}$ denoted the induced connection on $\operatorname{Ad} P$, $\operatorname{hol}(\varpi)$ *denotes the holonomy of the connection (which is well-defined up to conjugacy in G), and* Z *is the centraliser of a subgroup.*

Now since $\Gamma(\operatorname{ad} P)$ is the Lie algebra of $\Gamma(\operatorname{Ad} P)$, we obtain the following corollary:

Corollary 2.20 *In particular the dimension of the stabiliser is given by:*

$$\dim \operatorname{stab}_{\mathscr{G}(P)}(\varpi) = \dim \ker \nabla^\varpi_{\mathrm{Ad}} = \dim \ker d_\varpi.$$

In other words, a connection is reducible if its holonomy group has a non-discrete centraliser. We draw attention at this point to conflicting terminology. A connection can be reducible to a sub-bundle $P' \subset P$ while still being irreducible in the above sense. Consider a manifold with finite non-abelian fundamental group $\pi_1(M) \subset \mathrm{SU}(2)$. Then there is a flat connection on the trivial $\mathrm{SU}(2)$-bundle over M with holonomy $\pi_1(M)$. It can be checked that $\mathrm{Z}(\mathrm{hol}) = \{\pm 1\}$, but clearly this connection is reducible to a $\pi_1(M)$-bundle over M. We conclude our preliminary discussion of the moduli space by describing its local structure at an irreducible connection:

Proposition 2.21 (Coulomb charts) *Let $\varpi \in \mathscr{C}(P)$ be irreducible. Then there is a neighborhood $0 \in U \subset \Omega^1(M, \operatorname{ad} P)$ and a chart of $\mathscr{B}^*(P)$ around $[\varpi]$ given by:*

$$\left\{\tilde{\varpi} = \varpi + a : \delta_\varpi a = 0, a \in U\right\} \mapsto \mathscr{B}^*(P)$$

This result follows from **Hodge theory** (cf. A.7). It can be shown that the tangent space of $\mathscr{G}(P) \cdot \varpi$ is exactly given by $\operatorname{im}(d_\varpi : \Omega^0 \to \Omega^1)$. Thus if we want to find a slice of the action, meaning pick a single connection out of each equivalence classe, we need to fix an orthogonal complement of $\operatorname{im} d_\varpi$. But given the choice of a Riemannian metric, such a complement can explicitly be given by $\ker(\delta_\varpi : \Omega^1 \to \Omega^0)$, which is the motivation for the result above. It has as a consequence that there is an isomorphism $\ker \delta_\varpi \simeq T_{[\omega]}\mathscr{B}^*(P)$ via the projection map $\mathscr{C}^*(P) \to \mathscr{B}^*(P)$, so it is warranted to call a vector in $\ker \delta_\varpi$ **horizontal**. Correspondingly, the **vertical** vectors, which get mapped to 0 under the projection are exactly the ones in $\operatorname{im} d_\varpi$.

Flat connections revisited

We indicated above that there was a relation between flat connections on a principal G-bundle and representations of the fundamental group into G. Denote by $\mathscr{R}(P) \subset \mathscr{B}(P)$ the subset of gauge equivalence classes of flat connections. This is a well-defined notion, since the curvature of a connection F is gauge-invariant. We can now state the precise result:

Theorem 2.22 *Let P be a principal G-bundle over a manifold M. Then there is an isomorphism:*

$$\mathscr{R}(P) \simeq \operatorname{Hom}(\pi_1(M), G)/G,$$

which is given by the holonomy representation of a flat connection. Here G acts on the set $\operatorname{Hom}(\pi_1(M), G)$ by conjugation. This isomorphism restricts to:

$$\mathscr{R}^*(P) \simeq \operatorname{Hom}^*(\pi_1(M), G)/G,$$

where $\operatorname{Hom}^*(\pi_1(M), G)$ denotes the irreducible representations.

2.3.1 Moduli spaces on the tube $\mathbb{R} \times N$

We have seen in 2.10 that families of connection on a manifold N and connections in temporal gauge (w.r.t. some fixed trivialisation) on the tube $\mathbb{R} \times N$ correspond. In what follows we will explore this correspondence on the level of gauge equivalence classes. Fix trivializations of $P \to \mathbb{R} \times N$ and $Q \to N$. The lemma 2.11 implies that for this fixed trivialisation and any given connection $\varpi \in \mathscr{C}(P)$ there is a gauge transformation $\varphi \in \mathscr{G}(P)$ such that $\varphi^*\varpi$ is in temporal gauge. So for any class in $\mathscr{B}(P)$ there is a lift ϖ which is temporal. Moreover, any two such lifts ϖ, ϖ' are related by a gauge transformation $\varphi \in \mathscr{G}(P)$ such that $d\varphi[\partial t] = \partial t$, where ∂t is the horizontal lift of $(0, \partial t) \in \mathscr{X}(N \times \mathbb{R})$ w.r.t ϖ. This follows from the fact that $d\varphi[\partial t]$ is also a horizontal lift of $(0, \partial t)$ w.r.t. ϖ. Now the gauge group of P is given as:

$$\mathscr{G}(P) \simeq C^\infty(\mathbb{R} \times N, G) \simeq C^\infty(\mathbb{R}, C^\infty(N, G)) \simeq C^\infty(\mathbb{R}, \mathscr{G}(Q)),$$

where we used the smooth exponential law (A.3). Seeing φ as a path in $\mathscr{G}(Q)$, the requirement $d\varphi[\partial t] = \partial t$ means exactly that the path is constant. Thus this procedure gives rise to a well-defined assignment:

$$\mathscr{B}(P) \to C^\infty(\mathbb{R}, \mathscr{C}(Q))/\mathscr{G}(Q).$$

It is now easy to see that this map is both injective (since we pass to the correct quotient on the right-hand side) and surjective by the gluing construction from 2.10.

Note however that $C^\infty(\mathbb{R}, \mathscr{B}(Q)) \simeq \mathscr{B}(P) \to C^\infty(\mathbb{R}, \mathscr{C}(Q))/C^\infty(\mathbb{R}, \mathscr{G}(Q)) \neq C^\infty(\mathbb{R}, \mathscr{C}(Q))/\mathscr{G}(Q)$, thus considering paths in $\mathscr{B}(Q)$ loses some information. Indeed, every constant path of gauge equivalence classes has a plethora of non-gauge equivalent lifts in $\mathscr{C}(P)$. However we can still get a grip on the path-space of $\mathscr{B}(Q)$ by fixing a way to uniquely lift them to $C^\infty(\mathbb{R}, \mathscr{C}(Q))$, and then proceeding as before. The main result in this respect is the following:

Proposition 2.23 *There is a one-to-one correspondence:*

$$\left\{ [\varpi] \in \mathscr{B}(P) : \iota_t(\delta_\varpi \mathrm{F}_\varpi) = 0 \right\} \leftrightarrow C^\infty(\mathbb{R}, \mathscr{B}(Q)).$$

Here ι_t denotes contraction by the vector field ∂_t.

Consider a path $\gamma : \mathbb{R} \to \mathscr{B}(Q)$. We say that a lift $\tilde{\gamma} : \mathbb{R} \to \mathscr{C}(Q)$ of γ is **horizontal** if:

$$\delta_{\tilde{\gamma}(t)} \frac{d}{dt} \tilde{\gamma}(t) = 0.$$

In other words, a path is horizontal if its velocity vector field is in the distribution given by the local Coulomb slices. Away from irreducibles, these slices define a pre-connection on the infinite-dimensional fibre bundle $\mathscr{C}^*(Q) \to \mathscr{B}^*(Q)$. And similar to the theory of connections on finite-dimensional bundles, it is always possible to find such a lift, which is moreover unique up to the choice of a lift of $\gamma(0)$. It can then be shown that the existence and uniqueness of this lift extends even to paths which contain reducible connections. Thus we have the following inclusion:

$$C^\infty(\mathbb{R}, \mathscr{B}(Q)) \xhookrightarrow{\text{hor}} C^\infty(\mathbb{R}, \mathscr{C}(Q))/\mathscr{G}(Q) \simeq \mathscr{B}(P).$$

Now to finish up the proof of the proposition 2.23, we need to identify the subset of horizontal paths in $\mathscr{B}(P)$. Thus assume that $\varpi \in \mathscr{C}(P)$ is a connection which is in temporal gauge and has come from a horizontal path $\omega_t \in \mathscr{C}(Q)$. Note that by 2.13 the fact that ϖ is in temporal gauge means that:

$$\iota_t \mathrm{F}_{\mathbf{A}} = \iota_t\left(d\mathbf{A} + \frac{1}{2}[\mathrm{F}_{\mathbf{A}}, \mathrm{F}_{\mathbf{A}}]\right) = \iota_t\,(d\mathbf{A}) = \frac{dA_t}{dt}.$$

Passing to the connection forms again we obtain $\iota_t \mathrm{F}_\varpi = \frac{dt\omega_t}{dt}$. Thus horizontality is equivalently described by the condition:

$$0 = \delta_{\omega(t)}\frac{d}{dt}\omega(t) = \delta_{\omega(t)}(\iota_t \mathrm{F}_\varpi) = \iota_t(\delta_\varpi \mathrm{F}_\varpi),$$

which concludes the proof.

2.4 Maxwell's equations via principal bundles

After having reviewed the theory of principal bundles, we will now revisit the Maxwell equations from the previous section in a more general setting, namely that of $\mathrm{U}(1)$ gauge theory. Consider therefore a principal $\mathrm{U}(1)$-bundle $P \to M$ over a Lorentzian four-manifold (M, g). We were able to reduce the Maxwell equations to the set of two equations for a two-form $F \in \Omega^2(M, \mathbb{R})$:

$$\boxed{dF = 0, \quad \delta F = J}$$

We already indicated that over contractible neighbourhoods the form F admits a local potential, i.e. a form $A|_U \in \Omega^1(U)$ such that $F|_U = dA|_U$, but that for more complicated manifolds one needs to be careful when one glues together different local potentials. We will now see that a connection $\varpi \in \mathscr{C}(P)$ provides an elegant framework to incorporate these compatibility conditions. So how to connections generalize potentials? Since $\mathfrak{u}(1) = i\mathbb{R}$, the curvature of ϖ is an element of $i\Omega^2_{\mathrm{U}(1)}(P, \mathbb{R})$, or equivalently $i\,\mathrm{F}_\varpi \in \Omega^2(M, \mathrm{ad}\,P)$. Recall that the Bianchi identity for the curvature of ϖ gives us $d\Omega = [\Omega, \varpi]$. From this, together with the definition of the induced connection on the adjoint bundle we obtain:

$$d_\varpi \mathrm{F}_\varpi = \left(d\,(\mathrm{F}_\varpi)^\sharp + \varpi \wedge_{\mathrm{ad}} (\mathrm{F}_\varpi)^\sharp\right)^\flat = (d\Omega + [\varpi, \Omega])^\flat = ([\Omega, \varpi] + [\varpi, \Omega])^\flat = 0,$$

which is something very similar to the the first Maxwell equation! In fact, since $\mathrm{U}(1)$ is abelian, its adjoint representation is trivial, and thus $d_\varpi = d$ corresponds to the usual de Rham differential in some fixed trivialisation. Hence $d_\varpi \mathrm{F}_\varpi = 0$ reads in this trivialization as $d\,\mathrm{F}_\varpi = 0$ for some $\mathrm{F}_\varpi \in \Omega^2(M)$, which is exactly the first Maxwell equation! Thus it is natural to ask for solutions of the form $i\,\mathrm{F}_{\mathbf{A}}$ to the Maxwell equations, since the first Maxwell equation is always satisfied for geometric reasons (similar to how working with a potential also automatically solves the first

Maxwell equation). In this analogy $i\mathbf{A}$ and $i\varpi$ take on the role of the potential from last chapter, and $i\,\mathrm{F}_{\mathbf{A}}$ and $i\,\mathrm{F}_c\,g$ become the generalization of the electromagnetic tensor. The new equations now read for $\varpi \in \mathscr{C}(P)$, where we do not require the bundle to be trivial:

$$\boxed{d_\varpi i\,\mathrm{F}_\varpi = 0, \quad \delta_\varpi i\,\mathrm{F}_\varpi = iJ} \tag{2.3}$$

This is a system of equations that reduces to the Maxwell equations in the case $G = \mathrm{U}(1)$, but can a priori be written down for any connection on any principal bundle. Now let $\varphi \in \mathscr{G}(P)$ be a gauge transformation of our principal bundle and let $\psi \in C^\infty(M, \mathrm{U}(1))$ be the associated map. If we assume that M is simply connected, we can write $\psi(p) = e^{if(p)}$. By the transformation law 2.7, we see that under this gauge transformation, the connection potential transforms as:

$$\varphi^*(\mathbf{A}) = \mathrm{ad}_\varphi\,\mathbf{A} - d(e^{if})e^{-if} = \mathbf{A} - (idf)e^{if}e^{-if} = \mathbf{A} - idf.$$

Thus we have $i\varphi^*(\mathbf{A}) = i\mathbf{A} + df$, which is exactly the form of gauge transformation we introduced in the first chapter. Here we used the fact that ad is trivial for $G = \mathrm{U}(1)$. In fact because of the commutativity of $\mathrm{U}(1)$, the utility of this gauge theory is somewhat limited, as the space of solutions will always be an affine space (since the equations are linear), and thus not much information can be obtained this way. Next chapter we will investigate this set of equations in the case $G = \mathrm{SU}(2)$, where the equations are non-linear, and thus admit more varied and interesting solution spaces.

Now let us also quickly review the Lagrangian formalism. We would like to interpret the second equation as $dS(\varpi) = 0$ for some functional defined on $\mathscr{C}(P)$. We propose the following:

$$\mathrm{YM}(\varpi) = \frac{1}{2}\int_{M,g} \langle \mathrm{F}_\varpi, \mathrm{F}_\varpi \rangle_{\mathrm{ad}\,P}.$$

Since $\mathrm{F}_{\varpi+ta} = \mathrm{F}_\varpi + td_\varpi a + O(t^2)$, the proof of 1.2 shows that indeed the critical points of this functional are the solutions to the **Yang-Mills** PDE 2.3.

So to conclude, the critical points of a functional, which also are solutions to a PDE, give insight into properties of the base space. From the next chapter on we will try to apply this general principle to the more complicated situation of the classification of three-manifolds.

2.5 The group $\mathrm{SU}(2)$

So far we have done everything in full generality, however it is important to realize that the gauge group we will be considering in the following chapters will mostly be $\mathrm{SU}(2) = \left\{A \in \mathbb{C}^{2\times 2} : \det(A) = 1, AA^* = \mathrm{id}\right\}$, a group with a number of special properties.

First, as a smooth manifold, SU(2) is diffeomorphic to the three-sphere S^3. An explicit diffeomorphism is given by:

$$\Phi : S^3 \subset \mathbb{C}^2 \to \mathrm{SU}(2) \subset \mathbb{C}^{2\times 2}; \quad (\alpha, \beta) \mapsto \begin{pmatrix} \alpha & -\bar{\beta} \\ \beta & \bar{\alpha} \end{pmatrix}$$

Sometimes it is also useful to think of SU(2) as the group of unit quaternions Sp(1), which allows for the interpretation of any SU(2)-bundle as a quaternionic vector-bundle. Going back to the gauge group $\mathscr{G}(P)$ of a trivial principal bundle, this diffeomorphism to S^3 allows for some explicit computations. Let us look at the case when the base manifold is three-dimensional. In the following we will usually denote three-manifolds by N and principal SU(2)-bundles over N by Q. We see that since Q is trivial we have $\mathscr{G}(Q) \simeq C^\infty(N, S^3)$, and in particular that $\pi_0(\mathscr{G}(Q)) \simeq [N, S^3] \simeq \mathbb{Z}$, given by the degree (e.g. by the Pontryagin-Thom construction [6]). We define the **degree of a gauge transformation** $\varphi \in \mathscr{G}(Q)$ to be the degree of the associated map $\psi \in C^\infty(N, S^3)$. We will see in the next chapter that the Chern-Simons functional is invariant under small gauge transformations, but makes discrete jumps under big gauge transformations, so the difference is worthwhile to keep in mind.

The Lie algebra $\mathfrak{su}(2)$

The Lie algebra of SU(2) is given by the traceless skew-adjoint matrices

$$\mathfrak{su}(2) = \left\{ A \in \mathbb{C}^{2\times 2} : \mathrm{tr}(A) = 0, A + A^* = 0 \right\}.$$

A basis useful for computation is given by:

$$\sigma_1 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \ \sigma_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \ \sigma_3 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

These are the so called **Pauli matrices**. Finally, let us give a tractable expression of the Maurer-Cartan form $\varphi_{MC} \in \Omega^1(\mathrm{SU}(2), \mathfrak{su}(2))$. Note that since SU(2) is parallelizable, its tangent bundle is trivial, and spanned by the fundamental vector fields corresponding to the σ_i. With this in mind one can directly verify that the one-form $\varphi_{S^3} \in \Omega^1(S^3, TS^3)$ given locally by:

$$\varphi_{S3} = dx^1 \otimes \partial_1 + dx^2 \otimes \partial_2 + dx^3 \otimes \partial_3 \tag{2.4}$$

corresponds to the pull-back of φ_{MC} under the diffemorphism Φ, where we implicitly identified the forms with values in $\mathfrak{g}$ with the forms with values in TS^3.

Chapter 3

Yang-Mills and Chern-Simons

We will now investigate the Yang-Mills functional YM we derived last chapter as a generalisation of the electromagnetic action. As general critical points of this functional are difficult to construct, we will concentrate on the local minimisers in the special case of four-manifolds. They will solve a simpler PDE, the instanton equation, whose Fredholm properties over compact domains we recall. We will then prove that instantons over Riemannian tubes correspond to gradient flow lines of a functional defined on the space of connections of the cross-section, the Chern-Simons functional CS. Its critical points are exactly the flat connections, and the local behaviour around them is related to the cohomology of the cross-section, twisted by the flat connection. Finally, we will express the Fredholm index of the deformation operator for the instanton equation via the spectral flow of the Hessian of the Chern-Simons functional.

3.1 Yang-Mills Theory

In this section we will be looking at an extension of the electromagnetic action from the first chapter to arbitrary (potentially non-abelian) gauge groups, the **Yang-Mills** functional. Let therefore (M^n, g) be a compact Riemannian manifold and G be a compact Lie group. Let $P \to M$ be a principal G-bundle.

Definition 3.1 (Yang-Mills functional) *Let $\varpi \in \mathscr{C}(P)$ be a smooth connection one-form. We define the action functional:*

$$\mathrm{YM}(\varpi) = \frac{1}{2} \int_{M,g} \langle \mathrm{F}_\varpi, \mathrm{F}_\varpi \rangle_{\operatorname{ad} P} = \frac{1}{2} \|\mathrm{F}_\varpi\|^2_{\operatorname{ad} P}.$$

For $G = \mathrm{U}(1)$, this is exactly the functional which describes electromagnetism on general $\mathrm{U}(1)$-bundles, as we have seen last chapter.

Remark 3.2 *In Yang-Mills theory, compactness of the base manifold is necessary to assure convergence of the integral defining* YM, *and compactness of the structure group G allows to define an* ad*-invariant metric on G which can then descend to* ad P.

Now we can apply the same treatment to YM as we did for $\mathscr{S}_{em}$, namely we are interested in the critical points of this functional, the so called **Yang-Mills connections**. But in contrast to the discussion of $\mathscr{S}_{em}$, we will immediately put everything on a sound analytical foundation. The correct way to achieve this is to work with connections and gauge transformations of an appropriate Sobolev regularity (cf. A.1 and A.2). To be precise we will work with connections of class $W^{3,2}$ and gauge transformations of class $W^{4,2}$ in the four-dimensional theory. We then have the following result:

Proposition 3.3 (Extension of YM) *The functional* YM *on a four-dimensional manifold extends to a smooth map $\mathscr{C}^{3,2}(P) \to \mathbb{R}$, which will also be denoted by* YM.

Proof The curvature defines a smooth map between Banach manifolds:

$$\mathrm{F} : \mathscr{C}^{3,2}(P) \to W^{2,2}(\Omega^2(M, \mathrm{ad}\, P)).$$

Let us first discuss why the map is continuous. We locally have $\mathrm{F}_{\mathbf{A}} = d\mathbf{A} + \frac{1}{2}[\mathbf{A}, \mathbf{A}]$. Now the exterior derivative is a bounded linear operator between Sobolev spaces $W^{k,p} \to W^{k-1,p}$ by proposition A.8. Since we furthermore have the Sobolev embedding $W^{3,2} \hookrightarrow W^{2,4}$, the quadratic term $\frac{1}{2}[\mathbf{A}, \mathbf{A}]$ will also be in $W^{2,2}$, and the assignment is continuous. Now onto smoothness. For this note that for $v \in T_{\varpi}\mathscr{C}^{3,2}(P) \simeq W^{3,2}(\Omega^1(M, \mathrm{ad}\, P))$ we have:

$$\mathrm{F}_{\varpi+tv} = \mathrm{F}_{\varpi} + t d_{\varpi} v + t^2 v \wedge v.$$

Thus by fixing a connection, we see that the curvature is a quadratic function, and by construction all its coefficients are $W^{2,2}$. So its first and second derivative are continuous maps $W^{3,2} \to W^{2,2}$, and since these are the only non-vanishing derivative, we have shown that F is a smooth map. Now smoothness of YM follows from the fact that the additional data of an inner product on $W^{2,2}(\Omega^2)$ is also a bounded assignment. □

Note however that the precise regularity is of little importance, as we are only interested in the critical points of YM, which can be shown to be automatically smooth in the context we need them. The Yang-Mills functional is also invariant under gauge transformations.

Proposition 3.4 (Gauge invariance of YM) *The functional* YM *on a four-dimensional manifold descends to a smooth functional on $\mathscr{B}^{*,3,2}(P) = \mathscr{C}^{*,3,2}(P)/\mathscr{G}^{4,2}(P)$.*

Proof The fact that $\mathrm{YM}(\varphi^*\varpi) = \mathrm{YM}(\varpi)$ follows from proposition 2.18, since we already have $\mathrm{F}_{\varpi} = \mathrm{F}_{\varphi^*\varpi}$. Smoothness can then be verified using the Coulomb charts from proposition 2.21. □

We can recover the analogue of the Maxwell equations for the Yang-Mills functional. The proof is essentially the same as the proof of theorem 1.2.

Proposition 3.5 (Yang-Mills equation) *A connection $\varpi \in \mathscr{C}^{3,2}(P)$ (or a class $[\varpi] \in \mathscr{B}^{*,3,2}(P)$) is a critical point of* YM *exactly when the* ***Yang-Mills equations*** *hold:*

$$\boxed{\begin{cases} d_\varpi \, \mathrm{F}_\varpi = 0 \\ \delta_\varpi \, \mathrm{F}_\varpi = 0 \end{cases}} \tag{3.1}$$

Notice that ϖ appears both in the curvature and in the exterior derivative, in addition to the fact that the expression of the curvature already contains a non-linear term. This makes the Yang-Mills equations a lot harder to solve than the linear Maxwell equations.

3.1.1 Instantons on four-manifolds

In this section we are going to look at a special class of solutions of the Yang-Mills equations which appear for bundles in four dimensions. Let (M^4, g) be a smooth Riemannian four-manifold without boundary. It comes with an Hodge-star operator $\star : \Omega^\bullet(M) \to \Omega^{4-\bullet}(M)$, which in particular maps 2-forms to 2-forms. Since $\star^2 = \mathrm{id}$ on $\Lambda^2 T_p M$ is a fibrewise involution and $\star$ acts isometrically, the two-forms split into the ± 1-eigenspaces:

$$\Omega^2(M) = \Omega^2_+(M) \oplus \Omega^2_-(M).$$

Elements of the positive eigenspace are called **self-dual**, whereas elements of the negative eigenspace are called **anti-self-dual**. Every form $\alpha \in \Omega^2(M)$ admits a unique decomposition $\alpha = \alpha^+ + \alpha^-$, into a self-dual and an anti-self-dual part. Moreover, the splitting readily extends to differential forms with coefficients, for instance:

$$\Omega^2(M, \mathrm{ad}\, P) = \Omega^2_+(M, \mathrm{ad}\, P) \oplus \Omega^2_-(M, \mathrm{ad}\, P).$$

Hence one can ask whether the curvature form F_ϖ is (anti)-self-dual. We call connections with anti-self-dual curvature **instantons**. They satisfy the **instanton equation**:

$$\boxed{\mathrm{F}^+_\varpi = 0 \Leftrightarrow \mathrm{F}_\varpi + \star\, \mathrm{F}_\varpi = 0} \tag{3.2}$$

Note that the instanton equation has a high degree of symmetry. Since the curvature is gauge-invariant, if some connection in a gauge equivalence class is an instanton, then every connection in that class is. Thus the instanton equation can be seen as a condition on $\mathscr{B}^*(P)$. Furthermore, the instanton equation is conformally invariant, since the Hodge operator on two-forms is invariant under conformal change $f \in C^\infty(M)$ of the metric, i.e. $\star_g = \star_{fg}$. This happens because the contribution of f in the volume form and the Hodge-operator cancel exactly in this special case.

Let $\varpi \in \mathscr{C}(P)$ be an instanton. It is then indeed a solution to the original Yang-Mills equations as well, since:

$$\delta_\varpi \mathrm{F}_\varpi = \star d_\varpi \star \mathrm{F}_\varpi = - \star d_\varpi \mathrm{F}_\varpi = 0$$

by the Bianchi identity. The instanton equation thus provides a simpler way to construct Yang-Mills connections in the four-dimensional case. In fact it can be shown that instantons are exactly the minimizers of the Yang-Mills functional, thus the more complicated saddle-type critical points cannot be constructed this way. Furthermore, this is clearly particular to dimension four, since in any other dimension the curvature is not middle-dimensional, thus it is not possible to speak of ASD curvature forms. Let us now take a closer look at the set of instantons.

3.1.2 The deformation operator

The self-dual part of the curvature $\mathrm{F}^+ : \mathscr{C}(P) \to \Omega^2_+(M, \operatorname{ad} P)$ can be seen as a smooth map between Banach manifolds of regularity $W^{3,2} \to W^{2,2}$. In this setting the set of instantons is then then exactly the pre-image of the zero section, $(\mathrm{F}^+)^{-1}(0)$. We would like to investigate the local properties of this set, such as whether it is a (preferably finite-dimensional) manifold and if so of which dimension. More precisely, we hope that F^+ is a **Fredholm map**, which means that its differentials are all **Fredholm operators** (see A.3 for more details), as this implies that the inverse image of a regular value is a smooth *finite dimensional* manifold.

Let thus $\varpi \in \mathscr{C}(P)$ be an instanton over M. We consider the **linearized instanton equation**, which for $v \in T_\varpi \mathscr{C}(P) \simeq \Omega^1(M, \operatorname{ad} P)$ reads:

$$d\,\mathrm{F}^+_\varpi[v] = 0 \Leftrightarrow d^+_\varpi v = 0.$$

Here $d^+_\varpi = \pi_+ \circ d_\varpi$ is the exterior derivative on $\Omega(M, \operatorname{ad} P)$ composed with the orthogonal projection on the space of self-dual forms. Is d^+_ϖ Fredholm? Unfortunately not. The problem comes from the the gauge invariance of the Yang-Mills functional. Since a neighborhood of $\mathrm{id} \in \mathscr{G}(P)$ is diffeomorphic to the infinite dimensional space $\Gamma(\operatorname{ad} P)$, any instanton is surrounded by an infinite-dimensional family of gauge-equivalent instantons. Thus the set of instantons cannot be finite-dimensional. For **partial differential operators** on manifolds (such as d^+_ϖ) one condition which implies the Fredholm property is **ellipticity** (see A.5). Notice however that

$$\dim T^*M \otimes \operatorname{ad} P = 4 \dim G > 3 \dim G = \dim \Lambda^2 T^*M \otimes \operatorname{ad} P,$$

so that d^+_ϖ cannot be elliptic (which was of course to be expected), but rather is **under-determined**. This can be remedied by considering the instanton equation modulo gauge equivalence, i.e. by considering F^+ as a map

$$\mathrm{F}^+ : \mathscr{B}^*(P) \to \Omega^2_+(M, \operatorname{ad} P).$$

By working in charts given by the Coulomb gauge, we can see the derivative of this new mapping as an operator extending $d\,\mathrm{F}^+$:

$$\begin{aligned} D_\varpi : \Omega^1(M, \mathrm{ad}\, P) &\to \Omega^0(M, \mathrm{ad}\, P) \oplus \Omega^2_+(M, \mathrm{ad}\, P) \\ \alpha &\mapsto (-\delta_\varpi \alpha, d\,\mathrm{F}^+_\varpi[\alpha]). \end{aligned}$$

This operator is called the **deformation operator** and describes the first-order behaviour of gauge equivalence classes of instantons around an instanton ϖ. We will in the discussion of this operator and others like it often abbreviate:

$$D_\varpi : \Omega^1(\mathrm{ad}\, P) \to \Omega^{0,2^+}(\mathrm{ad}\, P)$$

if no confusion is possible. The principal symbol of D_ϖ is given by:

$$\sigma_{D_\varpi}(\xi_i dx_i) = -\iota_{\xi_i \partial x^i}(\cdot) + \frac{1}{2}(\xi \wedge \cdot + \star\xi \wedge \cdot) : T^*M \otimes \mathrm{ad}\, P \to \mathrm{ad}\, P \oplus \Lambda^2 T^*M \otimes \mathrm{ad}\, P$$

which can readily be checked to be an isomorphism for every $\xi \neq 0$. Just notice that the dimension of the target and source now agree, and that for $v \in \ker \sigma_{D_\varpi}(\xi_i dx_i)$ we must have:

$$\left\{ \begin{array}{rl} \frac{1}{2}(\xi \wedge v + \star\xi \wedge v) & = 0 \\ v(\xi^\sharp) & = 0 \end{array} \right. \Leftrightarrow \left\{ \begin{array}{rl} v & = \lambda\xi, \lambda \in \mathbb{R} \\ v(\xi^\sharp) & = 0 \end{array} \right. \Leftrightarrow \xi = 0.$$

Thus D_ϖ is elliptic, which makes the instanton equation equation on the moduli space a non-linear elliptic equation. In particular this implies that D_ϖ is Fredholm (cf. A.22), so has a finite-dimensional kernel. Thus, if 0 is a regular value of F^+, then the space of instantons modulo gauge equivalence is a finite-dimensional manifold $(\mathrm{F}^+)^{-1}(0) \subset \mathscr{B}^*(P)$!

3.1.3 Instantons on tubes

In the development of the Floer theory we will be interested in the instanton-equation over non-compact manifolds. We now introduce this generalisation by first looking at **Riemannian tubes**, i.e. manifolds of the form $M = \mathbb{R} \times N$ with a product metric $dt^2 + g^2$. Suppose $P \to M$ is a principal $\mathrm{SU}(2)$-bundle, which is necessarily trivial (as M has the homotopy type of a three-dimensional cell complex, cf. proposition B.13). What form does the instanton equation take over tubular manifolds such as M? Denote by $\star$ the Hodge-star operator on M and by $\star_3 = \star_g$ the Hodge-star on N. The two are related by the following lemma:

Lemma 3.6 *Let $\varphi \in \Omega^2(M)$. Then in local coordinates it is of the following form:*

$$\varphi(\mathbf{x}, t) = \psi_t(\mathbf{x}) + dt \wedge \chi_t(\mathbf{x}),$$

with $\psi_t \in \Omega^2(N)$ and $\chi_t \in \Omega^1(N)$ smooth families. We then have:

$$\star\varphi = \star_3 \chi_t + dt \wedge \star_3 \psi_t.$$

Proof Fix a local coordinate system (t, x^1, x^2, x^3) on M, where t is the standard positively oriented coordinate. We can assume that (dx^1, dx^2, dx^3) is a positive local orthonormal frame of T^*N. Then (dt, dx^1, dx^2, dx^3) is also positively oriented. Every 2-form φ on M is locally of the form:

$$\varphi(\mathbf{x}, t) = a_{i,j}(\mathbf{x}, t)dx^i \wedge dx^j + b_k(\mathbf{x}, t)dt \wedge dx^k$$

Set $\psi_t = a_{i,j}(\cdot, t)dx^i \wedge dx^j$ and $\chi_t = b_k(\cdot, t)dx^k$ for the first part of the proposition. By definition, the Hodge operator on the product satisfies:

$$\star(dx^i \wedge dx^j) = dt \wedge dx^k \text{ and } \star\,(dt \wedge dx^k) = dx^i \wedge dx^j$$

where $j \equiv i+1 \mod 3$ and $k \equiv i+2 \mod 3$. For the Hodge operator on N we have the following identities:

$$\star_3(dx^i \wedge dx^j) = dx^k \text{ and } \star_3\, dx^k = dx^i \wedge dx^j$$

In particular this implies that

$$\star(dx^i \wedge dx^j) = dt \wedge \star_3(dx^i \wedge dx^j) \text{ and } \star\,(dt \wedge dx^k) = \star_3 dx^k.$$

Applied to φ this yields:

$$\star\varphi = dt \wedge \star_3\psi_t + \star_3\chi. \qquad \square$$

Corollary 3.7 *The anti-self-dual forms are exactly the forms which are locally in the shape* $\psi_t \wedge dt + \star_3\psi_t$*.*

Proof Locally, an anti-self-dual form is of the form $\varphi(\mathbf{x}, t) = \psi_t(\mathbf{x}) + dt \wedge \chi_t(\mathbf{x})$. And hence by the previous lemma:

$$\psi_t(\mathbf{x}) + dt \wedge \chi_t(\mathbf{x}) = \varphi = -\star\varphi = -\star_3\chi_t - dt \wedge \star_3\psi_t$$

from which follows: $\psi_t = -\star_3\chi_t$, and so

$$\varphi(\mathbf{x}, t) = \psi_t + \star_3\psi_t \wedge dt. \qquad \square$$

Let us return to the instanton equation on the tube. Let $\varpi \in \mathscr{C}(P)$ be a connection on a four-manifold. Since P is trivial, it admits a global connection potential $\mathbf{A} \in \Omega^1(M, \mathfrak{g})$ with respect to some trivialisation. Furthermore we can assume that $\mathbf{A}$ is in temporal gauge by proposition 2.11. Via the correspondance 2.10 we get a family of connection potentials on N, denoted $A(t)$. Since $\operatorname{ad} P$ is trivial, F_ϖ corresponds to a $\mathfrak{g}$-valued two-form, which is given by proposition 2.13 as:

$$\mathrm{F}_{\mathbf{A}} = d\mathbf{A} + \frac{1}{2}[\mathbf{A}, \mathbf{A}].$$

Since we assumed ϖ to be in temporal gauge we have $\mathbf{A} = \sum_{i=1}^{3} A_i dx^i$, and hence:

$$\begin{aligned} \mathrm{F}_{\mathbf{A}}(\partial t, \partial x_i) &= \frac{\partial}{\partial t} A_i \\ \mathrm{F}_{\mathbf{A}}(\partial x_i, \partial x_j) &= \frac{\partial}{\partial x^i} A_j - \frac{\partial}{\partial x^j} A_i + [A_i, A_j] = \mathrm{F}_{A(t)}(\partial x_i, \partial x_j). \end{aligned}$$

Here $\mathrm{F}_{A(t)}$ is the curvature of the connection $A(t)$ on the 3-dimensional slice $\{t\} \times N$. The instanton condition demands that F_{ϖ} and hence also $\mathrm{F}_{\mathbf{A}}$ be of the form $\psi_t \wedge dt + \star_3 \psi_t$, which considering the above, implies in temporal gauge that:

$$-\star_3 \frac{\partial}{\partial t} A_i dx^i = \mathrm{F}_{A(t)} \Leftrightarrow \frac{\partial}{\partial t} A(t) = -\star_3 \mathrm{F}_{A(t)} . \tag{3.3}$$

So if we choose an appropriate trivialisation of our bundle with respect to which an instanton ϖ is in temporal gauge, the instanton equation has the form of a time-evolution for $A(t)$. What can we say about the instanton equation on the level of the moduli spaces? What is the correct three-dimensional description of instantons? Fix a trivialisation of P and let $[\varpi] \in \mathscr{B}^*(P)$ be an instanton, where ϖ is in temporal gauge. We have seen in 2.3.1 that there is a 1-1 correspondence between equivalence classes of horizontal connections on P (meaning $\iota_t(\delta_{\varpi} \mathrm{F}_{\varpi}) = 0$) and paths in $\mathscr{B}(Q)$. Since instantons are clearly horizontal (as they satisfy the Yang-Mills equation $\delta_{\varpi} \mathrm{F}_{\varpi} = 0$), $[\varpi]$ corresponds to a unique path $\gamma \in C^{\infty}(\mathbb{R}, \mathscr{B}(Q))$, and from any horizontal lift $\gamma_{hor} \in C^{\infty}(\mathbb{R}, \mathscr{C}(Q))$ of γ the instanton can be reconstructed. By the above discussion we know that the connection potentials A_t of $\gamma_{hor}(t)$ satisfy the evolution equation 3.3 above. Since the assignment $\omega \leftrightarrow A$ is a linear isomorphism compatible with the correspondence $\mathrm{F}_{\omega} \leftrightarrow \mathrm{F}_A$ we obtain the following equation on the level of connections:

$$\boxed{\frac{\partial}{\partial t} \omega(t) = -\star_3 \mathrm{F}_{\omega(t)} .}$$

Finally note that the vector field $-\star_3 \mathrm{F}_{\omega}$ on $\mathscr{C}(Q)$ is both horizontal (meaning $-\star_3 \mathrm{F}_{\omega} \in \ker \delta_{\omega} \simeq T_{[\omega]} \mathscr{B}^*(Q)$) and compatible with the isomorphism $\ker \delta_{\omega} \simeq \ker \delta_{\varphi^* \omega}$. It therefore descends to define a vector field on $\mathscr{B}^*(Q)$. We hence have identified the instantons in the three dimensional picture:

Proposition 3.8 *Integral curves on $\mathscr{B}^*(Q)$ of the vector field $-\star_3 \mathrm{F}_{\omega}$ correspond to gauge equivalence classes of instantons on $\mathbb{R} \times N$ with the product metric.*

3.2 Chern-Simons Theory

Let N be a compact orientable three-manifold and Q an $\mathrm{SU}(2)$-bundle over N. This bundle is necessarily trivial for topological reasons (see example B.13). In this section we will define a functional whose critical points are the flat connections on Q and whose negative gradient flow equation is the evolution equation discussed in the previous section. We give different viewpoints, each highlighting an distinct aspect of the functional.

Since the three-dimensional oriented cobordism ring Ω_3^{SO} is trivial (cf. section B.3), N bounds a compact four-manifold M. Moreover, M has a collar diffeomorphic to $N \times [0, \varepsilon)$ as the cobordism is smooth. Since Q is trivial, it can be seen as the pullback via the inclusion $\iota : N \hookrightarrow M$ of the trivial bundle $P = M \times \mathrm{SU}(2)$. We have the following extension result for connections over Q:

Proposition 3.9 *Let $\omega \in \mathscr{C}(Q)$ Then there is a connection ϖ on P such that $\omega = \iota^*\varpi$, and which is the product connection on a collar of M.*

Proof Decompose $M = U \cup V$, where $U \simeq N \times [0, 1)$ is a collar and V is an open set not intersecting the boundary. Define a connection ϖ_U on U as the pullback of ω under the projection map $p : U \simeq N \times [0, 1) \to N$, and choose a connection ϖ_V on V arbitrarily. A standard partition of unity argument allows us to glue both of these connections together to form $\varpi \in \mathscr{C}(P)$ without disturbing the connection ϖ_U on $N \times [0, \frac{1}{2})$. As $\iota^* p^* = (p \circ \iota)^* = \mathrm{id}^*$ we have $\iota^*\varpi = \iota^* p^* \omega = \omega$. □

Let $\pi : P \to M$ be an $\mathrm{SU}(2)$-bundle over a compact base manifold (potentially with boundary), with a connection $\varpi \in \mathscr{C}(P)$. Using Chern-Weil theory (see section B.2.1) one can define the **Chern forms** $c_i(\varpi) \in \Omega^{2i}(M)$, which give insight into the twistedness of the principal bundle. On four-manifolds the only non-vanishing ones are c_0, c_1 and c_2. However from their formulae we see that c_0 is a fixed constant, and that for special unitary bundles c_1 always vanishes, see example B.16. Hence the only truly interesting one is:

$$c_2(\varpi) = \frac{1}{8\pi^2} \int_M \mathrm{tr}(\mathrm{F}_\varpi \wedge \mathrm{F}_\varpi).$$

As it is a top-dimensional form it can be integrated, and for closed M its integral is in fact equal to the topologically defined **second Chern number** $c_2(P) \in \mathbb{Z}$ associated to the principal bundle, independent of the connection. It turns out that this topologically defined number is enough to determine the isomorphism type of an $\mathrm{SU}(2)$-bundle over a four-manifold (see B.13). The natural question to ask is then: what about manifolds with boundary? If having no boundary implies integrality, what influence does the boundary have on the possible Chern numbers? It need no longer be an integer, however there is the following restriction:

Proposition 3.10 *Let $\omega \in \mathscr{C}(Q)$ be a smooth connection one-form on a three-manifold. Let ϖ_1, ϖ_2 be extensions over some (not necessarily diffeomorphic) four-manifolds M_1, M_2 as above. Then:*

$$\int_{M_2} c_2(\varpi_2) - \int_{M_1} c_2(\varpi_1) \in \mathbb{Z}.$$

Proof As M_1, M_2 are tubular in neighbourhoods of their common boundary N, the glued manifold $M = \overline{M_1} \cup_N M_2$ can be given a smooth structure in a canonical way from the charts on N. The same holds true for the trivial bundles over M_1 and M_2, which can be fit together to form a (potentially non-trivial) bundle $P \to$

M. Furthermore, since the connections ϖ_1 and ϖ_2 are product connections in a neighborhood of N, they fit smoothly together to yield a connection ϖ on P. Finally, as M is closed we arrive at:

$$\begin{aligned}\int_{M_2} c_2(\varpi_2) - \int_{M_1} c_2(\varpi_1) &= \int_{M_2} c_2(\varpi_2) + \int_{\overline{M}_1} c_2(\varpi_1) \\ &= \int_M c_2(\varpi) = c_2(P) \in \mathbb{Z}. \qquad \square\end{aligned}$$

Thus the boundary condition of a fixed connection on ∂M uniquely defines the fractional part of the Chern-Weil integral for c_2. Using this insight, we can define the **Chern-Simons functional** for connections over three-manifolds.

Definition 3.11 (Chern-Simons Functional) *Let $\omega \in \mathscr{C}(Q)$ be a smooth connection one-form. Let ϖ be any extension as above. We define:*

$$\mathrm{CS}(\omega) = \int_M c_2(\varpi) \in \mathbb{R}/\mathbb{Z}$$

So $\mathrm{CS}(\omega)$ is the reduction $\bmod 1$ of $\int_M c_2(\varpi)$. Note that from this definition it is immediately clear that $\mathrm{CS}(\vartheta) = 0$ for any trivial connection ϑ, since we always can extend Q to a trivial bundle over M in such a way that the extended connection is trivial as well, and thus has vanishing curvature. Let us now derive a different formula for CS which allows for easier manipulation. Let for this a trivialisation of Q be fixed. Hence we can associate to each connection $\omega \in \mathscr{C}(Q)$ a connection potential $A \in \Omega^1(N, \mathfrak{su}(2))$.

Proposition 3.12 (Explicit expression in terms of trivialization) *We have for ω and A as above:*

$$\mathrm{CS}(\omega) = \frac{1}{8\pi^2}\int_N \mathrm{tr}(A \wedge dA + \frac{2}{3} A \wedge A \wedge A) \in \mathbb{R}/\mathbb{Z}.$$

This is independent of the trivialization chosen.

Proof Consider the manifold $M = [0,1] \times N$ with boundary $\partial M = N \times \{0\} \sqcup \bar{N} \times \{1\}$, and trivialise $P \to M$ so that over the boundary the trivialisation of P and Q agree. Define the connection $\varpi = \Theta + tA$ on M (where we consider $tA \in \Omega^1_G(P, \mathfrak{g})$, and Θ is the trivial connection), and note that over the ends it pulls back to the three-dimensional trivial connection ϑ (for $t = 0$) and ω (for $t = 1$) respectively. Hence:

$$\mathrm{CS}(\omega) = \mathrm{CS}(\omega) - \mathrm{CS}(\vartheta) = \int_M c_2(\varpi).$$

Recall that $\mathrm{F}_{\Theta + tA} = \mathrm{F}_\Theta + d(tA) + t^2 A \wedge A = dt \wedge A + tdA + t^2 A \wedge A$. Here d denotes the exterior derivative on $\Omega^\bullet(M, \mathrm{ad}\,P) \simeq \Omega^\bullet(M, \mathfrak{g})$, which is just the usual de Rham differential for $\mathfrak{g}$-valued forms. Plugging this into the formula for the Chern form we obtain:

$$
\begin{aligned}
8\pi^2 c_2(\varpi) &= \operatorname{tr}(\mathrm{F}_{\Theta+tA} \wedge \mathrm{F}_{\Theta+tA}) \\
&= \operatorname{tr}\Big(dt \wedge A \wedge dt \wedge A + t(dt \wedge A \wedge dA + dA \wedge dt \wedge A) \\
&+ t^2(2dt \wedge A \wedge A \wedge A + dA \wedge dA) + t^3(A \wedge A \wedge dA + dA \wedge A \wedge A) \\
&+ t^4(A \wedge A \wedge A \wedge A)\Big) \\
&= \operatorname{tr}\Big(t(dt \wedge A \wedge dA + dA \wedge dt \wedge A) + t^2(2dt \wedge A \wedge A \wedge A)\Big) \\
&= 2\operatorname{tr}\Big(t(dt \wedge A \wedge dA) + t^2(dt \wedge A \wedge A \wedge A)\Big).
\end{aligned}
$$

The constant term vanishes since $dt \wedge dt = 0$, and the cubic and quartic terms do since they are four-forms on a three-dimensional slice $\{t\} \times N$. The manipulation in the last line is due to the identity $\operatorname{tr}(ABC) = \operatorname{tr}(BCA)$. Now, using Fubini we can integrate the dt-factor:

$$
\begin{aligned}
\int_M c_2(\varpi) &= \frac{1}{4\pi^2}\int_N \int_0^1 \operatorname{tr}\Big(t(dt \wedge A \wedge dA) + t^2(dt \wedge A \wedge A \wedge A)\Big) \\
&= \frac{1}{4\pi^2}\int_N \operatorname{tr}\left(\frac{1}{2}A \wedge dA + \frac{1}{3}A \wedge A \wedge A\right) \\
&= \frac{1}{8\pi^2}\int_N \operatorname{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right).
\end{aligned}
$$

□

We have shown that the real-valued integral $\frac{1}{8\pi^2}\int_N \operatorname{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right)$ is a lift of the circle-valued Chern-Simons functional, and from our treatment this lift seems to depend on the choice of trivialisation. This is in fact the case. We already know that trivial connections are zeroes of the circle-valued functional and thus the expression above must be integral whenever A corresponds to a trivial connection. For instance if $A = 0$ then the expression is zero. However if this connection (that was induced from the trivialisation of Q chosen) undergoes a big gauge transformation the value of the integral does in fact jump by a non-zero integer amount. Consider a gauge transformation coming from a map $g : N \to \mathrm{SU}(2)$. Then it transforms as $g^*0 = g0g^{-1} + g^*(\varphi_{MC}) = g^*(\varphi_{MC})$, and so:

$$
\int_N \operatorname{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) = \deg(g)\int_{\mathrm{SU}(2)} \operatorname{tr}\left(\varphi_{MC} \wedge d\varphi_{MC} + \frac{2}{3}\varphi_{MC} \wedge \varphi_{MC} \wedge \varphi_{MC}\right)
$$

Now due to the Maurer-Cartan equation $d\varphi_{MC} + \varphi_{MC} \wedge \varphi_{MC} = 0$ (since the Maurer-Cartan form is a connection form of the principal bundle $\mathrm{SU}(2) \to \{*\}$), we further reduce this to:

$$
\int_N \operatorname{tr}\left(A \wedge dA + \frac{2}{3}A \wedge A \wedge A\right) = \frac{-\deg(g)}{3}\int_{\mathrm{SU}(2)} \operatorname{tr}\left(\varphi_{MC} \wedge \varphi_{MC} \wedge \varphi_{MC}\right)
$$

An explicit computation using the expression 2.4 leads to the identity:

$$\operatorname{tr}(\varphi_{MC} \wedge \varphi_{MC} \wedge \varphi_{MC}) = 2\operatorname{tr}(\sigma_1^2 + \sigma_2^2 + \sigma_3^2) d\operatorname{vol}_{\mathrm{SU}(2)} = -12 d\operatorname{vol}_{\mathrm{SU}(2)}.$$

Here σ_i denote the Pauli matrices from 2.5 that span $\mathfrak{su}(2)$ and $d\operatorname{vol}_{\mathrm{SU}(2)}$ denotes the volume form induced by the diffeomorphism $S^3 \to \mathrm{SU}(2)$, meaning in particular that:

$$\int_{\mathrm{SU}(2)} d\operatorname{vol}_{\mathrm{SU}(2)} = \operatorname{vol}(S^3) = 2\pi^2.$$

This gives us the result:

$$\frac{1}{8\pi^2} \int_N \operatorname{tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) = \frac{-\deg(g)}{24\pi^2}(-12)(2\pi^2) = \deg(g).$$

We have thus proved that the lift to $\mathbb{R}$ of the Chern-Simons functional makes integral jumps depending on the degree of the transition map between different trivial connections. A slight extension of this result can be used to show gauge invariance of the functional.

Proposition 3.13 (Gauge invariance) *Let $\omega \in \mathscr{C}(Q)$ and $\varphi \in \mathscr{G}(Q)$. Then we have for any real-valued lift of* CS*:*

$$\mathrm{CS}(\varphi^*\omega) - \mathrm{CS}(\omega) = \deg\varphi.$$

Proof Consider a tube $[0,1] \times N$, with a connection $\omega \in \mathscr{C}(Q)$ over the first end and a connection $\varphi^*\omega$ in the same gauge class over the second end, and let ϖ be an adapted connection over the tube. Thus $\mathrm{CS}(\omega) - \mathrm{CS}(\varphi^*\omega) = \int_{[0,1]\times N} c_2(\varpi)$. However notice that we can form a quotient bundle over $S^1 \times N$ by identifying $p \in P|_{\{1\}\times N}$ with $\varphi(p) \in P|_{\{0\}\times N}$, and that ϖ descends to a smooth connection on the quotient bundle P. Since the new base space is closed we thus have that $\int_{[0,1]\times N} c_2(\varpi) = c_2(P)$ is the second Chern number obtained by gluing a bundle over $[0,1] \times N$ via a gluing map of degree $\deg\varphi$. However this same Chern number is obtained if we start with trivial connections over N, for which the explicit computation above shows that $c_2(P) = \deg\varphi$, which completes the proof. □

As a consequence of this result, for a loop $\gamma : S^1 \to \mathscr{B}(Q)$, we have that $\deg(\mathrm{CS} \circ \gamma) = \deg\varphi$. We can use this to illuminate the topology of $\mathscr{B}(Q)$ a bit further. Consider two loops $\gamma_0, \gamma_1 : S^1 \to \mathscr{B}^*(Q)$ based at the same class $[\omega]$. We can lift them to horizontal paths $\tilde{\gamma}_i : [0,1] \to \mathscr{C}(Q)$ with the same starting point. From this it is clear that they are homotopic loops iff their end-points are related by a small gauge transformation. Indeed if they are joined by such a small gauge transformation then it is easy to join up their ends in $\mathscr{C}(Q)$, and a homotopy is readily found. However if their ends are related by a big gauge transformation, then by the above $\deg(\mathrm{CS} \circ \gamma_0) - \deg(\mathrm{CS} \circ \gamma_1) = \deg(\mathrm{CS}(\gamma_0 * \gamma_1^{-1})) \neq 0$, so they cannot be homotopic.

Thus we have shown that $\pi_1(\mathscr{B}^*(Q)) = \mathbb{Z}$. One can then use the fact that the subset of reducible connections in $\mathscr{B}(Q)$ is of infinite codimension, so that by cellular approximation $\pi_1(\mathscr{B}(Q)) = \pi_1(\mathscr{B}^*(Q)) = \mathbb{Z}$.

We now introduce a class of Riemannian manifolds that generalises the Riemannian tubes we have already considered, and will be central to the study of both instantons and gradient flow lines of the Chern-Simons functional.

Definition 3.14 (Tubular manifold) *A **tubular manifold** is a Riemannian manifold (M^n, g), such that there is a compact set $K \subset M$ and a finite number of open sets $U_i, i = 1, \dots, k$, such that $M = K \sqcup \bigsqcup_{i=1}^n U_i$ and, each U_i is isometric to $N_i \times (0, \infty)$ for some Riemannian $(n-1)$-manifolds N_i, which do not need to be the same.*

Being tubular is a condition on both the smooth topology of a manifold, in that the finite number of ends need to have a compact cross-section, as well as its Riemannian structure, since the ends are required to be isometric to tubes. They are a class of manifolds well suited to study CS.

It is a direct consequence of the definition of CS that it behaves additively under disjoint union, i.e. : $\mathrm{CS}_{N \cup N'}(\omega \oplus \omega') = \mathrm{CS}_N(\omega) + \mathrm{CS}_{N'}(\omega')$, from which we immediately obtain:

Corollary 3.15 *Let M be a tubular four-manifold with ends $N_i, i = 1, \dots, k$ and $\varpi \in \mathscr{C}(P)$ be connection, which restricts to ω^i over N_i. We then have:*

$$\int_M c_2(\varpi) = \sum_{i=1}^n \mathrm{CS}(\omega^i) \in \mathbb{R}/\mathbb{Z}.$$

Now that we have proven a number of algebraic results about CS, let us continue our investigation by studying its analytic properties as a functional between Sobolev spaces. For reasons explained in the appendix A, we will work on three-manifolds with connections of class $W^{1,4}$, altough there is some flexibility in choosing these parameters. It is important to note though that elements of $\mathscr{C}^{1,4}(Q)$ are continuous by the Sobolev embedding theorem.

Proposition 3.16 (Differentiability of CS, 1b) in [7]) *The functional* CS *extends to a C^2-functional:*

$$\mathrm{CS} : \mathscr{C}^{1,4}(Q) \to \mathbb{R}/\mathbb{Z}.$$

The proof is very similar to the corresponding proof for the Yang-Mills functional. The local formula from proposition 3.12 is used, and the function spaces are chosen so that all the algebraic operations form bounded operators.

Proposition 3.17 (Derivative of CS) *The derivative of* CS *is the one-form $d\,\mathrm{CS}$ which evaluates on a tangent vector $v \in T_\omega \Omega^1(\mathscr{C}(Q)) \simeq \Omega^1(N, \mathrm{ad}\, Q)$ as:*

$$d\,\mathrm{CS}(\omega)[v] = \frac{1}{4\pi^2} \int_M \mathrm{tr}(\mathrm{F}_\omega \wedge v). \tag{3.4}$$

Proof We compute, where we write v for both the form in $\Omega^1(N, \operatorname{ad} Q)$ and $\Omega^1(N, \mathfrak{g})$:

$$
\begin{aligned}
&8\pi^2 \frac{\partial}{\partial t}\Big|_{t=0} \mathrm{CS}(\omega + tv) \\
&= \frac{\partial}{\partial t}\Big|_{t=0} \int_N \operatorname{tr}\Big((A+tv)\wedge d(A+tv) + \frac{2}{3}(A+tv)\wedge(A+tv)\wedge(A+tv)\Big) \\
&= \frac{\partial}{\partial t}\Big|_{t=0} \Big(\mathrm{CS}(\omega) + \int_N t\cdot \operatorname{tr}(v\wedge dA + A\wedge dv) + 2(A\wedge A\wedge v) + O(t^2)\Big) \\
&= \int_N \operatorname{tr}(v\wedge dA + A\wedge dv + 2A\wedge A\wedge v) \\
&= 2\int_N \operatorname{tr}((dA + A\wedge A)\wedge v) \\
&= 2\int_N \operatorname{tr}(\mathrm{F}_A \wedge v) = 2\int_N \operatorname{tr}(\mathrm{F}_\omega \wedge v).
\end{aligned}
$$

Here we used that $0 = \int_N d\operatorname{tr}(A\wedge v) = \int_N \operatorname{tr}(dA\wedge v - A\wedge dv)$, and that $\operatorname{tr}(ABC) = \operatorname{tr}(BCA)$. □

Proposition 3.18 (Critical points of CS) *A smooth connection $\omega \in \mathscr{C}(Q)$ is a critical point of* CS *iff it is flat.*

Proof Critical points of CS are connections $\omega \in \mathscr{C}(Q)$ such that $d\,\mathrm{CS}(\omega) = 0$. For every tangent vector $v \in \Omega^1(N, \operatorname{ad} Q)$ we must have:

$$0 = d\,\mathrm{CS}(\omega)[v] = \frac{1}{4\pi^2}\int_M \operatorname{tr}(\mathrm{F}_\omega \wedge v). \tag{†}$$

Recall from 2.5 that $\mathfrak{su}(2)$ is generated by the Pauli matrices σ_i. For $u = a\sigma_1 + b\sigma_2 + c\sigma_3$ we can compute explicitly tr $(u\cdot\sigma_1) = -2a$ and tr $(u\cdot\sigma_3) = 2(-c + bi)$. So by carefully choosing the test-section v, we can derive from (†) that all the components of F_ω vanish. This then implies that ω is flat. If on the other hand ω is flat, then $\mathrm{F}_\omega = 0$ and so it is clearly critical. □

Recall that we denote the subset of flat connections in $\mathscr{C}(Q)$ by $\mathrm{R}(Q)$, and the its projection in $\mathscr{B}(Q)$ by $\mathscr{R}(Q)$. Thus the proposition above states that the set of critical points of CS on $\mathscr{C}(Q)$ is exactly equal to $\mathrm{R}(Q)$, and on $\mathscr{B}^*(Q)$ it is given by $\mathscr{R}^*(Q)$. In order to differentiate between connections and their gauge equivalence classes we use these two different notations.

Remark 3.19 *Our approach here was to define the Chern-Simons functional and compute its derivative from that definition. Since we are however mostly interested in* CS *for its critical points and its infinitesimal behaviour (like integral curves of its gradient vector field), we could have started with the differential one-form given by* $d\,\mathrm{CS}$ *that we already have discovered in the last section and tried to integrate it.*

This is indeed possible, as the space of connections $\mathscr{C}^{1,4}(Q)$ is a Banach space. By Poincaré's lemma for open subsets of Banach spaces (V.4.1 in [8]) we can thus find a *primitive of every closed differential* 1*-form* $\rho \in \Omega^1(\mathscr{C}^{1,4}(P))$*. This holds in particular for* $\rho = d\,\mathrm{CS}$*. For this one needs to show that the form 3.4 is in fact closed (which we now know because we have obtained it as a differential). This can however be done explicitly and without appealing to the existence of the* CS*-functional.*

Taking everything together we have proven so far, we can conclude this section by stating:

Theorem 3.20 *The Chern-Simons functional is a* C^2*-functional on* $\mathscr{B}^*(Q)$*, whose critical set is exactly the set of gauge equivalence classes of flat connections* $\mathscr{R}(Q)$*.*

3.3 Local behaviour of the Chern-Simons functional

Gradient of CS

It is not always possible to define a gradient vector field dual to the differential of a functional on an infinite dimensional manifold. This is due to the lack of compactness of the unit sphere in an infinite dimensional normed space. However for the Chern-Simons functional we can write down a gradient explicitly using the particular form of its differential. For $\omega \in \mathscr{C}(Q)$, we will consider the L^2-scalar product on $\Omega^1(N, \mathrm{ad}\, Q) \simeq T_\omega \mathscr{C}(Q)$. We can thus rewrite the derivative 3.4 of CS as:

$$d\,\mathrm{CS}(\omega)[v] = \int_N \mathrm{tr}(\mathrm{F}_\omega \wedge v) = \int_N \mathrm{tr}(v \wedge \star_3(\star_3 \mathrm{F}_\omega)) = \langle\!\langle \star_3 \mathrm{F}_\omega, v \rangle\!\rangle.$$

It is thus natural to define the L^2-gradient vector field of CS to be

$$\mathrm{grad}\,\mathrm{CS}(\omega) = \star_3 \mathrm{F}_\omega \in \Omega^1(N, \mathrm{ad}\, Q).$$

Definition 3.21 *We say that a smooth family of connections* $\omega(t) \in \mathscr{C}(Q)$ *is a* ***gradient-flow line*** *of* CS *if it satisfies the evolution equation:*

$$\frac{\partial}{\partial t}\omega(t) = -\,\mathrm{grad}\,\mathrm{CS}(\omega(t)). \tag{3.5}$$

Remark 3.22 *The vector field* $-\,\mathrm{grad}\,\mathrm{CS}$ *does not in fact determine a flow on the infinite dimensional manifold it is defined on, so gradient flow should be understood in a formal way.*

We have already encountered this vector field in proposition 3.8. Hence we are now able to describe $-\star_3 \mathrm{F}_\omega$ as the negative gradient vector field of the Chern-Simons functional. By horizontality and gauge-invariance it descends to define a vector field on the moduli space $\mathscr{B}^*(Q)$ and so we can state:

Proposition 3.23 *The flow lines of the Chern-Simons functional on the moduli space* $\mathscr{B}^*(Q)$ *correspond to equivalence classes of Yang-Mills instantons over the tube* $\mathbb{R} \times N$*.*

Hessian of CS

Next, we would like to understand the second order behaviour of CS at a critical point $[\omega] \in \mathscr{R}^*(Q)$. We will for analytical reasons continue to work on the level of connections $\mathscr{C}(Q)$. The corresponding results for the moduli space will then be obtained by restricting to $\ker \delta_\omega \subset \Omega^1(N, \operatorname{ad} Q) \simeq T_\omega \mathscr{C}(Q)$, which is isomorphic to $T_{[\omega]}\mathscr{B}^*(Q)$ via the projection. For the purpose of eventually defining a Morse homology of CS, we are mostly interested in the positive, negative and null space of its **Hessian** at a critical point $\omega \in \mathrm{R}(Q)$:

$$\begin{aligned} \operatorname{Hess}_\omega(\mathrm{CS}) : \Omega^1(N, \operatorname{ad} Q) \times \Omega^1(N, \operatorname{ad} Q) &\to \mathbb{R} \\ (v, w) &\mapsto \left.\frac{d}{dt}\right|_{t=0} (d\,\mathrm{CS}(\omega + tw)[v]) \end{aligned}$$

Notice that on the level of connections, we will also work with reducible critical points, even though the moduli space admits a singularity there. In fact we can define the Hessian formally for any $\omega \in \mathscr{C}(Q)$. We have the following relation between the gradient and the Hessian of CS, obtained by a direct computation:

$$\forall u, v \in \Omega^1(N, \operatorname{ad} Q) : \langle d \operatorname{grad} \mathrm{CS}(\omega)[v], w\rangle = \operatorname{Hess}_\omega(\mathrm{CS})[u, v].$$

In other words, to understand the Hessian as a bilinear form, it is enough to determine the spectral properties of $d \operatorname{grad} \mathrm{CS}$ as an operator

$$H_\omega = d \operatorname{grad} \mathrm{CS} : \Omega^1(N, \operatorname{ad} Q) \to \Omega^1(N, \operatorname{ad} Q),$$

which is given on $v \in \Omega^1(N, \operatorname{ad} Q)$ by the expression

$$H_\omega(v) = \left.\frac{d}{dt}\right|_{t=0} \star_3 \mathrm{F}_{A+tv} = \star_3 d_\omega v.$$

However H_ω is analytically speaking not well behaved, as it is not elliptic (for the same reason that the usual de Rham differential is not elliptic, namely that $v \mapsto dx \wedge v$ is not injective). This is why we usually consider the following operator instead:

$$L_\omega : \Omega^{0,1}(\operatorname{ad} Q) \to \Omega^{0,1}(\operatorname{ad} Q); \quad L_\omega(f, v) = \begin{pmatrix} 0 & -\delta_\omega \\ -d_\omega & \star_3 d_\omega \end{pmatrix} \begin{pmatrix} f \\ v \end{pmatrix}$$

Note that up to musical isomorphism $\star_3 d_\omega$ is a twisted version of the usual curl-operation on three dimensional manifolds. To be more precise the de Rham complex and the vector calculus operations interact in the following way:

$$\begin{tikzcd} \Omega^0(N) \arrow[r, "d"] \arrow[d, "\simeq"] & \Omega^1(N) \arrow[r, "d"] \arrow[d, "\natural"] & \Omega^2(N) \arrow[r, "d"] \arrow[d, "\natural \circ \star_3"] & \Omega^3(N) \arrow[d, "\star_3"] \\ C^\infty(N) \arrow[r, "\mathrm{grad}"] & \mathscr{X}(N) \arrow[r, "\mathrm{curl}"] & \mathscr{X}(N) \arrow[r, "\mathrm{div}"] & C^\infty(N) \end{tikzcd}$$

Here $\natural$ denotes the musical isomorphism $\natural : \Omega^1(N) \to \mathscr{X}(N)$ induced from the Riemannian metric. For $\omega \in \mathrm{R}(Q)$ flat we can define the **twisted de Rham complex** (cf. A.2) where d_ω takes the place of the usual de Rham differential d. The other vector calculus operations thus also get analogues in terms of d_ω and δ_ω. In this analogy this new operator is equivalent to:

$$\begin{pmatrix} 0 & -\operatorname{div} \\ -\operatorname{grad} & \operatorname{curl} \end{pmatrix}$$

The operator L_ω is thus essentially a restricted version of the Dirac operator $D = d_\omega + \delta_\omega$, which is elliptic, motivating the choice. Consider now the Hessian at a critical point $\omega \in \mathrm{R}(Q)$. Notice that $\ker H_\omega = \ker d_\omega \supset \operatorname{im} d_\omega$, the latter of which is complemented (via Hodge theory, cf.A.46) by $\ker \delta_\omega$. Since we have by flatness of ω that $\delta_\omega \star_3 d_\omega = \star_3 d_\omega^2 = 0$, we can think of H_ω as an operator from $\ker \delta_\omega$ to itself, without loosing any information. This is the Hessian on the moduli space! Let us decompose $\Omega^0 \oplus \Omega^1 = \Omega^0 \oplus \operatorname{im} d_\omega \oplus \ker \delta_\omega$. Written in matrix notation, L_ω then has the form:

$$\begin{pmatrix} 0 & -\delta_\omega & -\delta_\omega \\ -d_\omega & \star_3 d_\omega & \star_3 d_\omega \\ -d_\omega & \star_3 d_\omega & \star_3 d_\omega \end{pmatrix} = \begin{pmatrix} 0 & -\delta_\omega & 0 \\ -d_\omega & 0 & 0 \\ 0 & 0 & H_\omega|_{\ker d_\omega} \end{pmatrix},$$

where most of the entries vanish by flatness. This is due to the both the identities $d_\omega^2 = 0$ and $\delta_\omega^2 = 0$ and the decomposition of forms as $\Omega^1 = \operatorname{im} d_\omega \oplus \ker \delta_\omega$. If we define a new operator $S_\omega : \Omega^0 \oplus \operatorname{im} d_\omega \to \Omega^0 \oplus \operatorname{im} d_\omega$ as

$$S_\omega = \begin{pmatrix} 0 & -\delta_\omega \\ -d_\omega & 0 \end{pmatrix}$$

we then get the decomposition $L_\omega = S_\omega \oplus H_\omega$, thus our new operator L_ω is an extension of our original Hessian H_ω. It behaves better analytically:

Theorem 3.24 *Let ω be a flat connection. The operator L_ω is self-adjoint, elliptic and $L_\omega^2 = \Delta_\omega$.*

Proof First we show that L_ω is self-adjoint. Compute for $(f,\alpha), (g,\beta) \in \Omega^0 \oplus \Omega^1$:

$$\begin{aligned} \langle (f,\alpha), L_\omega(g,\beta)\rangle_{\Omega^0\oplus\Omega^1} &= \langle (f,\alpha), (-\delta_\omega\beta, -d_\omega g + \star_3 d_\omega\beta)\rangle_{\Omega^0\oplus\Omega^1} \\ &= \langle f, -\delta_\omega\beta\rangle_{\Omega^0} + \langle \alpha, -d_\omega g + \star_3 d_\omega \beta\rangle_{\Omega^1} \\ &= \langle -d_\omega f, \beta\rangle_{\Omega^1} + \langle -\delta_\omega\alpha, g\rangle_{\Omega^0} + \langle \star_3 d_\omega\alpha, \beta\rangle_{\Omega^1} \\ &= \langle L_\omega(f,\alpha), (g,\beta)\rangle_{\Omega^0\oplus\Omega^1}. \end{aligned}$$

Here we used the fact that $\star_3^2 = 1$, so that $\star_3 d_\omega$ is self-adjoint. Next, we compute:

$$L_\omega^2 = \begin{pmatrix} 0 & -\delta_\omega \\ -d_\omega & \star_3 d_\omega \end{pmatrix}^2 = \begin{pmatrix} \delta_\omega d_\omega & -\star_3 d_\omega^2 \\ -\star_3 d_\omega^2 & \delta_\omega d_\omega + d_\omega \delta_\omega \end{pmatrix} = \Delta_\omega,$$

where we need the flatness of ω in the form $d_\omega^2 = 0$. Finally, the operator is elliptic because its square $L_\omega^2 = \Delta_\omega$ is (cf. A.45). Indeed and thus $\sigma_{L_\omega}^2 = \sigma_{\Delta_\omega}$ is invertible, so that σ_{L_ω} is invertible as well. □

The operator $L_\omega : W^{1,2} \to L^2$ on suitable Sobolev spaces is thus a self-adjoint elliptic operator, and as such has real and discrete spectrum $\sigma(L_\omega) = \sigma(H_\omega) \cup \sigma(S_\omega)$. Note the unfortunate clash of notation: $\sigma_P(\xi)$ denotes the symbol of a partial differential operator, while $\sigma(P)$ denotes its spectrum, which are of course quite different concepts.

We will finish up the discussion of $\mathrm{Hess}(\mathrm{CS})$ by giving an interpretation of the kernel of L_ω in terms of the **twisted cohomology groups** $\mathrm{H}^*(N, d_\omega)$ **of** ω . Indeed this should not be surprising, as we have just shown that L_ω squares to the Hodge Laplacian.

Definition 3.25 *Let $\omega \in \mathrm{R}(Q)$ be a flat connection. Then we say that ω is:*

- ***acyclic**, if $\mathrm{H}^*(N, d_\omega) = 0$.*
- ***non-degenerate**, if $\mathrm{H}^1(N, d_\omega) = \mathrm{H}^2(N, d_\omega) = 0$.*

Note that by Poincaré duality for the twisted cohomology groups A.48, acyclicity is already achieved if $\mathrm{H}^0(N, d_\omega) = \mathrm{H}^1(N, d_\omega) = 0$, and and non-degeneracy if $\mathrm{H}^1(N, d_\omega) = 0$. These cohomology groups have distinct interpretations:

Proposition 3.26 *The connection $\omega \in \mathrm{R}(Q)$ has $\mathrm{H}^0(N, d_\omega) = 0$ iff it is irreducible.*

Proof Recall that the stabilizer of a connection is a Lie-subgroup of $\mathrm{SU}(2)$ with dimension $\dim \ker d_\omega \subset \Omega^0$ by 2.19. But $\ker d_\omega = \mathrm{H}^0(N, d_\omega)$, and ω is irreducible exactly when its stabilizer group is discrete, hence 0-dimensional. □

Proposition 3.27 *The connection $\omega \in \mathrm{R}(Q)$ has $\mathrm{H}^1(N, d_\omega) = 0$ iff it is is a non-degenerate critical point of* CS*, meaning that the Hessian on the moduli-space is non-degenerate.*

Proof The Hessian at a critical point $\omega \in \mathscr{R}(Q)$ is non-degenerate exactly when $\ker H_\omega|_{\ker \delta_\omega} = \ker \star_3 d_\omega \subset \ker \delta_\omega \subset \Omega^1$ is trivial. Even more, the tangent space to the set of flat connections in $\mathscr{B}(Q)$ at ω is exactly given by $\ker H_\omega$. This can for instance be seen by deriving $d\,\mathrm{CS}$ of a smooth curve of flat connection, for which the curvature will be constantly zero. Since $\star_3 : \Omega^2 \to \Omega^1$ is isomorphism, this kernel is given by

$$\ker H_\omega = \ker \delta_\omega \cap \ker d_\omega = (\operatorname{im} d_\omega)^\perp \cap d_\omega \simeq \mathrm{H}^1(N, d_\omega). \qquad \square$$

Thus we have recovered different geometric properties of the connection $[\omega]$ through the twisted cohomology groups $\mathrm{H}^0(N, \omega)$ and $\mathrm{H}^1(N, \omega)$. Notice that S_ω is defined on $\Omega^0 \oplus \operatorname{im} d_\omega$, thus any element $(f, \alpha) \in \ker S_\omega$ must have $\alpha = 0$, since $\alpha \in \ker \delta_\omega \cap \operatorname{im} d_\omega = \ker \delta_\omega \cap (\ker \delta_\omega)^\perp = \{0\}$, by the Hodge theorem. From this it is clear that $\ker S_\omega \simeq \ker d_\omega \simeq \mathrm{H}^0(N, \omega)$. We hence have:

$$\ker L_\omega = \ker S_\omega \oplus \ker H_\omega|_{\ker \delta_\omega} \simeq \mathrm{H}^0(N, \omega) \oplus \mathrm{H}^1(N, \omega)$$

Thus for an irreducible flat connections ω, the kernel of L_ω describes exactly the degree of degeneracy of the Hessian on the level of gauge equivalence classes.

Example 3.28 *Consider a trivial connection $\vartheta \in \mathscr{C}(Q)$. In this case the twisted de Rham complex reduces to the regular de Rham complex tensored with the Lie algebra* $\mathfrak{su}(2)$ *of* $\mathrm{SU}(2)$. *So we have:*

$$\mathrm{H}^k(N, d_\vartheta) \simeq \mathrm{H}^k(N) \otimes \mathfrak{su}(2).$$

In particular if N is connected, any trivial connection is reducible, with a 3*-dimensional stabilizer, since* $\dim(\mathrm{H}^k(N) \otimes \mathfrak{su}(2)) = 3$. *If N is a* ***homology sphere****, i.e. a manifold which has the same homology groups as S^3, then it is furthermore non-degenerate. If the manifold is not a homology sphere however, then the trivial connection cannot be non-degenerate for* CS.

The three-dimensional interpretation of D_ϖ

We have seen that the instanton equation on a tube can be recast as the negative gradient flow equation of the Chern-Simons functional on the cross-section. In this context we were able to translate back and forth between four-dimensional concepts, such as the second Chern number of the four-dimensional bundle $c_2(P)$ and the degree of a loop in $\mathscr{B}(Q)$. There should also be an relation between the deformation operator D_ϖ, which describes the infinitesimal behaviour of instantons, and some other operator, describing the infinitesimal behaviour of CS-flow-lines. In fact there is exactly such an expression in terms of the family $L_{\omega(t)}$. But in order to answer this question, we first need to be able to move from forms on $M = \mathbb{R} \times N$ to curves of forms on N. More precisely we have the following, where work with fixed trivialisations of P and Q:

Proposition 3.29

$$\begin{aligned}
\Omega^0(M, \operatorname{ad} P) &\simeq C^\infty(\mathbb{R}, \Omega^0(N, \operatorname{ad} Q)) \\
\Omega^1(M, \operatorname{ad} P) &\simeq C^\infty(\mathbb{R}, \Omega^0(N, \operatorname{ad} Q) \oplus \Omega^1(N, \operatorname{ad} Q)) \\
\Omega^2_+(M, \operatorname{ad} P) &\simeq C^\infty(\mathbb{R}, \Omega^1(N, \operatorname{ad} Q))
\end{aligned}$$

Proof The first isomorphism reduces to the statement that $C^\infty(\mathbb{R} \times N) \simeq C^\infty(\mathbb{R}, C^\infty(N))$, which we address in A.3. Let $\sigma \in \Omega^1(M, \operatorname{ad} P)$ be given. In local coordinates (t, x^1, x^2, x^3) it can we written as $\sigma(t, \mathbf{x}) = \sigma^0(t, \mathbf{x})dt + \sum_{i=1}^3 \sigma^i(t, \mathbf{x})dx^i$. By assumption, the assignments $t \mapsto \sigma^k(t, \cdot)$ are smooth, and thus the path of differential forms $t \mapsto (\sigma(t, \cdot), \sum_{i=1}^3 \sigma^i(t, \cdot)dx^i)$ is smooth. In the reverse direction, take $(f_t, \sum_{i=1}^3 a_t dx^i)$ and fit them together as $f_t dt + \sum_{i=1}^3 a_t dx^i$ to see bijectivity. Similarly, a path $\alpha_t \in \Omega^1(N, \operatorname{ad} Q)$ gives rise to a form $\alpha_t \wedge dt + \star_3 \alpha_t$ on the tube which by 3.7 is anti-self-dual. It is clear that all ASD forms can be obtained this way by the same lemma. □

We are now in a position to recover D_ϖ from the local behaviour of $L_{\omega(t)}$ along a path in $\mathscr{C}(Q)$.

Proposition 3.30 (D_ϖ on a tube) *Let $\omega_t \in \mathscr{C}(Q)$ be a smooth path of connections on a three-manifold N. Let ϖ be the corresponding connection on the tube. By the previous proposition D_ϖ is an operator:*

$$D_\varpi : C^\infty(\mathbb{R}, \Omega^{0,1}(N, \operatorname{ad} Q)) \to C^\infty(\mathbb{R}, \Omega^{0,1}(N, \operatorname{ad} Q)) \tag{3.6}$$

It is given by $(D_\varpi \xi)(t) = \dot{\xi}(t) + L_{\omega(t)}\xi(t)$.

Proof We have the following situation:

$$\begin{tikzcd} C^\infty(\mathbb{R}, \Omega^{0,1}(N, \operatorname{ad} Q)) \arrow[r, "D_\varpi"] \arrow[d, "\simeq"] & C^\infty(\mathbb{R}, \Omega^{0,1}(N, \operatorname{ad} Q)) \\ \Omega^1(\mathbb{R}\times N, \operatorname{ad} P)) \arrow[r, "D_\varpi^{old}"] & \Omega^{0,2^+}(\mathbb{R}\times N, \operatorname{ad} P)) \arrow[u, "\simeq"'] \end{tikzcd}$$

Let $f_t \in \Omega^0(N, \operatorname{ad} Q), \alpha_t \in \Omega^1(N, \operatorname{ad} Q)$ with corresponding $f dt + \alpha = f dt + \alpha^i dx^i \in \Omega^1(M, \operatorname{ad} P)$. Then we compute the action of $D_\varpi = (-\delta_\varpi) \oplus d_\varpi^+$. Here the indices i, j, k are always chosen so that (dx^i, dx^j, dx^k) is a positive basis and $\pi_+ : \Omega^2 \to \Omega^2_+$ is given by $\pi_+(\sigma) = \sigma + \star\sigma$.

$$\begin{aligned}
-\delta_\varpi(f dt + \alpha^i dx^i) &= \star d_\omega \star (f dt + \alpha^i dx^i) \\
&= \star d_\omega (f dx^1 \wedge dx^2 \wedge dx^3 + \alpha^i dt \wedge dx^j \wedge dx^k (-1)^i) \\
&= \star(\nabla_t f + \nabla_i \alpha^i) dt \wedge dx^1 \wedge dx^2 \wedge dx^3 \\
&= \nabla_t f + \nabla_i \alpha^i = \frac{d}{dt} f - \delta_{\omega(t)} \alpha_t \\
d_\varpi^+(f dt) &= \pi_+(\nabla_i f dx^i \wedge dt) \\
&= (-\nabla_i f)\pi_+(dt \wedge dx^i) \\
d_\varpi^+(\alpha^i dx^i) &= \pi_+(\nabla_t \alpha^i dt \wedge dx^i + \nabla_j \alpha^i dx^j \wedge dx^i + \nabla_k \alpha^i dx^k \wedge dx^i) \\
&= \nabla_t \alpha^i \pi_+(dt \wedge dx^i) - \nabla_j \alpha^i \pi_+(dt \wedge dx^k) + \nabla_k \alpha^i \pi_+(dt \wedge dx^j) \\
&= (\nabla_t \alpha^i - \nabla_k \alpha^j + \nabla_j \alpha^k)\pi_+(dt \wedge dx^i)
\end{aligned}$$

Since the last arrow sends $\pi_+(dt \wedge dx^i) \mapsto dx^i$ we obtain:

$$\begin{aligned}
d_\varpi^+(f dt) &= (-\nabla_i f)\pi_+(dt \wedge dx^i) \\
&\mapsto -\nabla_i f dx^i = -d_{\omega(t)} f_t \\
d_\varpi^+(\alpha^i dx^i) &= (\nabla_t \alpha^i - \nabla_k \alpha^j + \nabla_j \alpha^k)\pi_+(dt \wedge dx^i) \\
&\mapsto \frac{d}{dt}\alpha_t + (\nabla_j \alpha^k - \nabla_k \alpha^j) dx^i = \frac{d}{dt}\alpha_t + \star_3 d_{\omega(t)} \alpha(t)
\end{aligned}$$

Hence we have arrived at the following formula for $\xi(t) = (f_t, \alpha_t) \in C^\infty(\mathbb{R}, \Omega^{0,1}(N, \operatorname{ad} Q))$:

$$(D_\varpi \xi)(t) = \frac{d}{dt}\xi(t) + \begin{pmatrix} 0 & -\delta_{\omega(t)} \\ -d_{\omega(t)} & \star_3 d_{\omega(t)} \end{pmatrix} \begin{pmatrix} f_t \\ \alpha_t \end{pmatrix} = \frac{d}{dt}\xi(t) + L_{\omega(t)}\xi(t)$$

□

With this identification, we are able to fit the deformation operator D_ϖ into the more general framework of the **spectral flow** of a family of self-adjoint operators, such as the family $t \mapsto L_{\omega(t)}$. We give an introduction to the spectral flow in the appendix (see A.6). In order to apply the theory, we need to consider operators on suitable Banach manifolds, and not simply on the Fréchet spaces of smooth curves. We have already seen that $D_\varpi : W^{3,2} \to W^{2,2}$ is a smooth map in the compact setting, which motivates us to define the new deformation operator as follows:

$$D_\varpi : W^{1,2}(\mathbb{R}, W) \to L^2(\mathbb{R}, H)$$

where $W = W^{3,2}(\Omega^{0,1}(N, \operatorname{ad} Q))$ and $H = W^{2,2}(\Omega^{0,1}(N, \operatorname{ad} Q))$. We however need the additional assumption that $\omega(t)$ converges to fixed limits $\omega_\pm$ as $t \to \pm\infty$. To see that this is then indeed a well-defined operator, first note that $W \hookrightarrow H$ is a inclusion (which is furthermore compact), and thus $\frac{d}{dt}$ is well-defined already. Next, we need so show that for a smooth path $\omega(t) \in \mathscr{C}(Q)$ the assignment:

$$\begin{aligned} L_{\omega(\cdot)} : \mathbb{R} &\to \mathscr{L}(W, H) \\ t &\mapsto L_{\omega(t)} \end{aligned}$$

is a continuous curve in the Banach space $\mathscr{L}(W, H)$. If it is, then convergence to limits of $\omega(t)$ assures that the curves will have uniform bound on its operator norm, thus making the map $W^{1,2} \to L^2$ bounded. Now why is the curve $t \mapsto L_{\omega(t)}$ continuous? Indeed, every $L_{\omega(t)}$ separately is a bounded linear operator $W \to H$ by proposition A.8, which makes the map well-defined. Now consider for a moment $\omega(0)$ as a reference connection, so that $\omega(t) = \omega(0) + A(t)$, where $A(t) \in \Omega^1(N, \operatorname{ad} Q)$ is smooth by assumption. Then their corresponding covariant derivatives on $\operatorname{ad} Q$ are related by $\nabla^{\omega(t)} - \nabla^{\omega(0)} = A(t)$. From the graded Leibniz rule, it can be checked that in general degree:

$$(d_{\omega(t)} - d_{\omega(0)})(s \otimes \sigma) = A(t)s \wedge \sigma \in \Gamma(\operatorname{Hom}(\Lambda^\bullet \otimes \operatorname{ad} Q, \Lambda^{\bullet+1} \otimes \operatorname{ad} Q)).$$

This assignment is still smooth. From there we see that the curve $L_{\omega(\cdot)}$ is a smooth curve in $\Gamma(\operatorname{Hom}(\Lambda^{0,1} \otimes \operatorname{ad} Q, \Lambda^{0,1} \otimes \operatorname{ad} Q))$, and that the operator norm of $L_{\omega(t)} - L_{\omega(0)}$ is up to a constant bounded by the operator norm of wedging with $A(t)$, which is again bounded by a suitable Sobolev norm. Now since there is a continuous inclusion of Fréchet spaces $\Omega^1(N, \operatorname{ad} Q) \hookrightarrow W^{k,p}(\Omega^1(N, \operatorname{ad} Q))$ by A.11, we have:

$$\lim_{t\to 0} \|L_{\omega(t)} - L_{\omega(0)}\| \leq C \lim_{t\to 0} \|A(t) \wedge \cdot\| \leq C \lim_{t\to 0} \|A(t)\|_{W^{k,p}} \leq C \lim_{t\to 0} \|A(t)\|_{C^k} \to 0.$$

Chapter 4

Floer Homology

We will now introduce the Floer homology groups of a homology three-sphere using the analytical and geometric foundations from the previous chapters. We emphasise the analysis of Fredholm operators over tubular manifolds, and how it gives rise to the trajectory spaces and gluing results necessary to carry out the Morse homology of the Chern-Simons functional. Furthermore we will put effort into explaining why it is important to focus on homology three-spheres rather than more general manifolds. This is mostly because of the presence of reducible connections and the way we perturb in a degenerate situation. We conclude with the discussion of the extension of the Floer homology groups to a $(3+1)$ topological quantum field theory.

4.1 Motivation

Instanton Floer homology, i.e. the Morse homology of the Chern-Simons functional, is an interesting invariant of three-manifolds for a number of reasons. We will here consider two motivations, coming from mathematical gauge theory and classical Morse homology respectively.

4.1.1 Invariants in Gauge theory

The general procedure to define invariants of smooth manifolds through gauge theory has the following schematics. Consider a principal bundle $P \to M$ over a compact manifold with structure group G, and the space of gauge equivalence classes $\mathscr{B}(P)$, which is an infinite dimensional manifold away from irreducibles. Consider then a (generally non-linear) elliptic PDE on $\mathscr{B}(P)$, potentially defined using some auxiliary data. More precisely we are interested in its solution set. Generically this will be a finite dimensional (by ellipticity) submanifold of the moduli space. From here the strategies diverge, but the main idea is to use this new manifold to define novel smooth invariants of the base space. We give a few examples:

1. **Electromagnetism.** In this case the equations were in fact linear (since the non-linear commutator term in F_ω vanishes for $G = U(1)$), and the solution space was an affine space of a certain dimension. In this example we only recovered the second Betti number, so no new invariant.

2. **Classical Donaldson theory.** Here we consider the instanton equation on an $SU(2)$-bundle $P \to M$ over a smooth, simply connected four-manifold with negative definite intersection form, such that $c_2(P) = 1$. We will see later that index theory predicts the dimension of the moduli space W to be 5. This moduli space can then be shown to be (after suitably compactifying) a smooth cobordism between M, which appears as one boundary component of W, and some disjoint copies of $\mathbb{C}P^2$. This cobordism then greatly restricts the possible intersection forms; in fact it can be shown that only the standard form $n[-1]$ can appear for smooth manifolds. A proof of this result can be found in chapter 8 of the book by Donaldson and Kronheimer [9].

3. **Seiberg-Witten theory.** In this case a different set of equations (the Seiberg-Witten equations) is considered on $U(1)$-bundles over a four-manifold. The moduli space of solutions (the monopoles) is already compact, and if the set of reducible monopoles is of sufficiently high codimension, perturbing the auxiliary data used to write down the equations will avoid reducibles and in fact describe a cobordism between the solution set for any two auxiliary data sets along the pertubation. From this we see that the set of monopoles defines a homology class of the parameter space which is independent of the auxiliary data. The paramenter space is homotopy equivalent to $\mathbb{C}P^\infty$, so evaluation on the corresponding cohomology class in $H^*(\mathbb{C}P^\infty) \simeq \mathbb{Z}[x_2]$ will lead to numerical data, the Seiberg-Witten invariants. Note that this approach is somewhat similar to the finite dimensional theory of degree of a map, where for a smooth map $f : X \to Y$, the degree is computed in terms of the solution set of the equation $f(x) = y$ with y a regular value of f, and a perturbation y' of y gives rise to an oriented cobordism between $f^{-1}(y)$ and $f^{-1}(y')$ (see chapter 5 of [6]). For an exposition of the Seiberg-Witten equations on four-manifolds, see [10].

Instanton Floer homology fits into this picture with some alterations. Here we consider the space of instantons over non-compact tubes, which requires some additional assumptions in the form of boundary data, and considering the homology class of the set of solutions will reduce to counting points in $\mathscr{B}(P)$. There is hope to expect that the Morse homology of the Chern-Simons functional in particular is well-defined, since we have shown the equivalence between flow lines of CS and instantons. This allows to apply a result of Uhlenbeck (see 4.15) to show the crucial compactness up to broken trajectories that is necessary for Morse homology.

4.1.2 Morse Homology

We have seen that by considering the instanton equation over a tube, one is naturally inclined to consider the Morse homology of the Chern-Simons functional. But there is also reason to construct a Morse homology of CS coming from three-manifold topology. There is an invariant of compact three-manifolds called the **Casson invariant** $\lambda(N)$, and it is computed by algebraically counting the irreducible representations of $\pi_1(N)$ into $\mathrm{SU}(2)$ up to conjugacy. The signs are determined by considering them as intersections of oriented spaces (see p.54 in [11]). By 2.22 this is the same as counting the irreducible flat connections on a trivial $\mathrm{SU}(2)$-bundle over N. Thus the Casson invariant is an algebraic count of the critical points of CS. More precisely there is a function $\mu_{\mathrm{CS}} : \mathrm{Crit}(\mathrm{CS}) \to \mathbb{Z}_2$ such that:

$$\lambda(N) = \frac{1}{2} \sum_{[\omega] \in \mathrm{Crit}(\mathrm{CS})} (-1)^{\mu_{\mathrm{CS}}([\omega])}.$$

Given a Morse function $f : M \to \mathbb{R}$, where M is compact, there is a similar formula, where now $\mu : \mathrm{Crit}(f) \to \mathbb{N}$ is the Morse index. Namely we can compute the Euler characteristic of M as follows:

$$\chi(M) = \sum_{p \in \mathrm{Crit}(f)} (-1)^{\mu(f)}.$$

This identity can be explained through the existence of the Morse complex, which is generated by the critical points of f, and computes the homology of M. One is thus lead to conjecture that there should be a corresponding chain complex generated by the flat irreducible connections to explain the formula for the Casson invariant! In fact this is exactly the Floer chain complex we will see in this chapter, and in the same way that Morse homology is a more refined invariant than the Euler characteristic, Floer homology will be a more refined version of the Casson invariant.

4.2 Homology three-spheres

We will only define the instanton Floer homology groups for integral **homology three-spheres**, i.e. three-manifolds which have the same integral homology groups as the three-sphere. This is due to the fact that they only admit a single reducible flat connection, the trivial connection, and that this connection is non-degenerate for the Chern-Simons functional. Furthermore we will see that the set of irreducible flat connections of a homology sphere is finite under suitable conditions.

First, some general remarks. For N an oriented, connected three-manifold, being a homology three-sphere is equivalent to having $\mathrm{H}^1(N) = 0$ by Poincaré duality. Now by the Hurewicz theorem this is the case exactly when the abelianisation of the fundamental group $\pi_1^{ab}(N) := \pi_1(N)/[\pi_1(N), \pi_1(N)]$ is trivial. Here $[\pi_1(N), \pi_1(N)]$ is the **commutator subgrop**, which is generated by all elements of the form $aba^{-1}b^{-1}$

for $a, b \in \pi_1(N)$. Next, recall that on any manifold M, the moduli space of flat connections is homeomorphic to the space of space of representations modulo conjugacy of $\pi(M)$ by 2.22, so we can equally well analyse the space of representations to determine the properties of $\mathscr{R}(P)$ for some $\mathrm{SU}(2)$-bundle P over M. If M is compact, then it has a finitely presentable fundamental group, say:

$$\pi_1(M) = \langle a_1, \ldots, a_k \mid R_1, \ldots, R_l \rangle,$$

where a_i are the generators and R_i are the relations that must hold. A representation into $\mathrm{SU}(2)$ is the same as a choice of k elements $\rho(a_i) \in \mathrm{SU}(2)$ such that $R_j(\rho(a_1), \ldots, \rho(a_k)) = 1$. We can thus embed:

$$\begin{aligned} \mathrm{Hom}(\pi_1(M), \mathrm{SU}(2)) \simeq & \{(g_1, \ldots, g_k) \in (\mathrm{SU}(2))^k : R_j(\rho(a_1), \ldots, \rho(a_k)) = 1\} \\ & \subset (\mathrm{SU}(2))^k \end{aligned}$$

as a (real) algebraic subset of the compact variety $(\mathrm{SU}(2))^k$. In particular the space $\mathrm{Hom}(\pi_1(M), \mathrm{SU}(2))$ with the induced topology is always compact if the structure group is and $\pi_1(M)$ is finitely presentable. Next, as the quotient of a compact space by a compact group action, the space $\mathrm{Hom}(\pi_1(M), \mathrm{SU}(2)) / \mathrm{SU}(2)$ must also be compact, but can of course contain singularities if group action is not free. Let us now specialise to three dimensions. An orientable three-manifold N can be decomposed into two simpler pieces, so called handle-bodies M_i, which are glued along their common boundary, a Riemann surface Σ, to produce the three-manifold. This is a result very specific to dimension three. If Σ has genus g, the handle-bodies will have a free fundamental group on g generators, and thus $\mathrm{R}(M_i) = \mathrm{SU}^g(2)$, making $\mathscr{R}^*(M_i)$ a $(3g-3)$-dimensional open manifold. It can furthermore be shown that $\dim \mathscr{R}^*(\Sigma) = 6g - 6$. This is useful, because it allows to think of $\mathscr{R}^*(N)$ as an intersection:

$$\mathscr{R}^*(N) = \mathscr{R}^*(M_1) \cap \mathscr{R}^*(M_2) \subset \mathscr{R}^*(\Sigma).$$

For dimensional reasons this intersection will generically be an open manifold of dimension 0, however this is not necessarily the case, as non-transverse intersections can occur, consider the handlebodies themselves for instance. However, after suitable perturbation, the intersection set will be a 0-manifold. The space $\mathscr{R}(N)$ of all representations is compact as we have already seen, but in general we cannot say anything about the compactness properties of $\mathscr{R}^*(N)$. If we restrict to homology spheres however, we can say much more, as the following proposition shows:

Proposition 4.1 *Let N be a homology 3-sphere. Then the only reducible flat connection is the trivial connection.*

Proof Suppose that $\omega \in \mathrm{R}(N)$ is reducible, i.e. by 2.19 the stabiliser of ω in $\mathscr{G}(N)$ is bigger than $\{\pm 1\}$. The centraliser of any element $a \in \mathrm{SU}(2)$ that is not ± 1 can be seen in $\mathrm{SO}(3)$ as the set of all rotations which have the same axis. The lift of this set of rotations is diffeomorphic to S^1 in $\mathrm{SU}(2)$. Thus the centraliser of hol^ω must either be contained in some circle subgroup of $\mathrm{SU}(2)$, if hol^ω contains elements besides ± 1, or be equal to all of $\mathrm{SU}(2)$. Suppose that it is contained in a non-trivial circle subgroup (which is strictly bigger than $\{\pm 1\}$ by reducibility). Note then that the holonomy is contained in the centraliser of this circle group. But by what we have said so far, the holonomy itself must be contained in a circle subgroup then. From there is is clear that $\mathrm{Z}(\mathrm{hol}^{\omega}) \simeq S^1$. In any case, we see that the holonomy of a reducible connection has image in some abelian subgroup of $\mathrm{SU}(2)$. Now since N is a homology-sphere, we have

$$\pi_1(N) = [\pi_1(N), \pi_1(N)].$$

Thus in particular every element $c \in \pi_1(N)$ is of the form $aba^{-1}b^{-1}$ for some $a, b \in \pi_1(N)$. For the holonomy representation $\rho = \rho^{\omega} \in \mathrm{Hom}(\pi_1(N), \mathrm{SU}(2))$ of ω this implies that:

$$\rho(c) = \rho(aba^{-1}b^{-1}) = \rho(a)\rho(b)\rho(a^{-1})\rho(b^{-1}) = 1,$$

since the image of ρ is abelian. Thus any reducible flat connection is in fact trivial, because its holonomy is trivial. □

Using this one can show (see e.g. lemma 3.6 in [11]) that for homology three-spheres the space $\mathscr{R}^*(N)$ is itself compact. Indeed compactness can only be lost if a subsequence of irreducible connections were to converge to the trivial connection. However $\mathrm{H}^1(N) \simeq 0$ will imply that the intersection at the trivial connection ϑ will be transverse, so that ϑ is an isolated point in $\mathscr{R}(N)$. The upshot of this is that if we are in a situation where all the flat connections are non-degenerate (and thus isolated by the Morse lemma), we see that the intersection describing $\mathscr{R}^*(N)$ is transverse and thus that it is a 0-manifold. Hence under this non-degeneracy condition we have:

$$\boxed{|\mathscr{R}^*(N)| < \infty \text{ and } \mathscr{R}(N) = \mathscr{R}^*(N) \cup \{\vartheta\}.}$$

From this point onward we will therefore work exclusively with homology spheres.

4.3 Index Theory

4.3.1 Adapted bundles

In order to generalise Morse homology to the case of the Chern-Simons functional, we need a good notion of trajectory space of flow lines between critical points. In the usual Morse homology of closed manifolds, every flow line starts and ends at critical points, and the dimension of the moduli space is the difference of the Morse indices of these two points. In particular the moduli space of curves joining two given critical points is generically a smooth manifold with the same dimension for each connected component. However this does not work in our setting, since there are CS-flow lines between the same gauge equivalence classes of flat connections, but around which the moduli spaces have different dimensions. The source of this discrepancy is that there are homotopically non-trivial loops that can arise as instantons. So in addition to keeping track of the end-points, a suitable parameter space to index the trajectory spaces must keep track of the homotopy class of the instantons. The correct notion to use, which also extends to the general case of tubular manifolds is the **adapted bundle**, which is defined in the following way:

Definition 4.2 *Let M be a tubular manifold. An **adapted bundle** over M is a principal* $\mathrm{SU}(2)$*-bundle with the additional data of a fixed flat product connection over each end.*

We say that two adapted bundles are equivalent, if there is a bundle morphism which fixes the flat structure over the ends. Note that here it is important that the flat connection is fixed, not only its gauge equivalence class. Consider the example of the tubular manifold W with one end obtained by puncturing S^4, and stretching out a neighbourhood $\mathbb{D}^4 \setminus \{*\}$ of the puncture to a semi-infinite tube. Let ϑ be a trivial connection over the cross-section S^3. It induces a trivialisation of $P|_{\{t\}\times S^3}$ for a three-dimensional slice. We can then extend P via a clutching construction (see [12]) to a unique bundle $\hat{P}$ over the compactification $\hat{W} \simeq S^4$, where the trivialisation tells us how to match up W with a trivial bundle over $\mathbb{D}^4$. Which bundle will we obtain over S^4? To determine this note that by 3.9 we can find a connection $\varpi \in \mathscr{C}(P)$ which agrees with the trivial connection over the tube. In fact it is possible to find such an **adapted connection** which agrees with the flat ends for every adapted bundle. And note that ϖ extends to a unique connection $\hat{\varpi} \in \mathscr{C}(\hat{P})$ which does not introduce additional curvature, meaning that:

$$c_2(\hat{P}) = \int_M c_2(\varpi) \cong \mathrm{CS}(\vartheta) \in \mathbb{R}/\mathbb{Z}.$$

Note that $c_2(\hat{P})$ thus provides an $\mathbb{R}$-valued lift of $\mathrm{CS}(\vartheta)$, so that if we gauge transform ϑ by a big gauge transformation, we change the second Chern number of the bundle $\hat{P}$, and thus its isomorphism type. In fact we can achieve any desired isomorphism type in this manner. This argument can be generalised by fixing for each gauge equivalence class of a flat connection $[\omega] \in \mathscr{R}(N_i)$ over an end a manifold $K_{[\omega]}$ by which to cap off M, and an adapted connection on $K_{[\omega]}$ with the given flat structure ω.

Let us explain this on the example of a tube $M = \mathbb{R} \times N$. Fix an adapted connection $\varpi \in \mathscr{C}(P)$ compatible with the flat ends $\omega_\pm \in \mathrm{R}(N)$ of M. Let $K_\pm$ be compact manifolds such that $\partial K_\pm = N$, and let $R_\pm$ be some fixed trivial bundle over K with an adapted connection $\varpi_K^\pm$ that agrees with the flat connection over, say, the positive end of the tube. Gluing $K_\pm$ onto M on both ends (with the orientation reversed) leads to a smooth connection $\tilde{\varpi}$ over $\tilde{M} = K_- \sqcup M \sqcup K_+$ with:

$$\int_{\tilde{M}} c_2(\tilde{\varpi}) = \int_M c_2(\varpi) + \int_{K_+} c_2(\varpi_K^-) + \int_{K_+} c_2(\varpi_K^+).$$

Thus by the previous paragraph, if we change the flat structure on either end of M by a big gauge transformation, then the integral over M will change by one, thus altering the isomorphism type of $\tilde{P}$. Since we have already seen that:

$$\int_M c_2(\varpi) = \deg \mathrm{CS} \circ \omega_t,$$

the isomorphism type of $\tilde{P}$ does in fact encode the homotopy class of the loop ω_t. Now since the data of an adapted bundle suffices to construct $\tilde{P}$ uniquely, they too encode the homotopy class in addition to the start and end connection.

Example 4.3 *Consider the sphere S^4, which (if we remove the north pole) is conformally equivalent to $\mathbb{R}^4$. There is a five-parameter family of instantons on the* $\mathrm{SU}(2)$ *bundle over S^4 with $c_2(P) = 1$. On $\mathbb{R}^4$ they all arise from a single basic instanton $\varpi_0 \in \mathscr{C}(P)$ after translations and scaling. This instanton has curvature density $|\mathrm{F}_{\varpi_0}| = \frac{1}{(1+r^2)^2}$, where r is the radial component on $\mathbb{R}^4$. We can further consider* $\mathbb{R}^4 \setminus \{0\}$ *as conformally equivalent to the tube $\mathbb{R} \times S^3$ via the transformation*

$$T : \mathbb{R} \times S^3 \to \mathbb{R}^4, (t, \sigma) \mapsto (e^t \sigma),$$

so that the basic instanton corresponds to a CS *gradient flow line in $\mathscr{B}^*(S^3)$. Since the only flat connection on S^3 is the trivial one ϑ, it must join ϑ to itself. This instanton has curvature density:*

$$\mathrm{F}_{\varpi_0}(t, \sigma) = |dT|^2 \, \mathrm{F}_{\varpi_0}(e^t \sigma) = e^{2t} \frac{1}{(1 + e^{2t})^2} = \frac{1}{(e^t + e^{-t})^2} = \frac{4}{\cosh^2(t)}.$$

Since in this case we can cap of both ends of the tube with trivial bundles that do not introduce further curvature, we must have:

$$\deg \mathrm{CS} \circ \omega_t = \int_{\mathbb{R} \times S^3} c_2(\varpi_0) = c_2(P) = 1.$$

Thus this instanton is an example of a generator of the fundamental group of $\mathscr{B}(Q)$. Through the five-parameter family of instantons on $\mathbb{R}^4$ we see that the local structure of the trajectory space at this instanton is that of a manifold of dimension 5. *This is different from the local structure around the constant instanton, which is zero-dimensional. Thus the homotopy class really does impact the shape of the moduli space!*

4.3.2 Fredholm theory

We have seen that the right way to describe the trajectory spaces of the Chern-Simons functional is through adapted bundles, and we explicitly gave the example of an instanton where the dimension of the moduli space was easily guessed. However to proceed in more generality, we need to develop the Fredholm properties of the operator D_{ϖ}. To this end, we first recall the situation for compact manifolds, before moving on to the case of tubes. There is a general theory for tubular manifolds for which all the same results hold.

We have already seen that in the compact case, the ellipticity of the operator D_ϖ implies that it is Fredholm. Its index can be computed in terms of its symbol and topological invariants of the underlying manifold (by the Atiyah-Singer Index theorem A.24). Explicitly we have the following result (see [1], p.10):

$$\operatorname{ind} D_\varpi = 8c_2(P) - 3(b_0 - b_1 + b_2^+). \tag{4.1}$$

Here b_i denote the Betti numbers of M, and b_2^+ is the dimension of a maximal positive definite subspace of $\mathrm{H}^2(M)$ of the intersection form.

Example 4.4 *Consider the instanton from example 4.3 on the* $\mathrm{SU}(2)$*-bundle over* S^4 *with* $c_2(P) = 1$*. From the index formula 4.1 we get:*

$$\operatorname{ind} D_\varpi = 8 \cdot 1 - 3(1 - 0 + 0) = 5,$$

which corresponds to the 4 *possible infinitesimal translations on* $\mathbb{R}^4$*, as well as the generator for the central scaling.*

Now in order to adapt the theory to tubular manifolds, we need to prove the Fredholm property for tubes. However by looking at the example of $D_\lambda = \frac{d}{dt} - \lambda : W^{1,2}(\mathbb{R}) \to L^2(\mathbb{R})$ for $\lambda \in \mathbb{R}$, we see that this cannot be done in a straightforward manner for all operators over the tube. The operator is Fredholm with index zero except when $\lambda = 0$, in which case it is not Fredholm. The problem is that in L^2, one can approximate arbitrarily well functions with non-zero integral by functions which do have zero integral. See also the remark after proposition A.22. In the case of $D_\varpi = \frac{d}{dt} - L_{\omega(t)}$ this failure to be Fredholm manifests as the limiting elliptic operators $L_{\omega(\pm\infty)}$ having a kernel, and must be treated carefully. Let us thus first consider the case where $L_{\omega(\pm\infty)}$ does not have a kernel, i.e. where the limiting connections are acyclic.

The acyclic case

Let P be an adapted bundle with acyclic ends over a tubular manifold M, and let $\varpi \in \mathscr{C}(P)$ be a connection that over every end converges smoothly to the chosen flat connection. An example would be an adapted connection, which is equal to the flat structure on every slice, or, as we will later see, an instanton.

Theorem 4.5 (D_ϖ is Fredholm) *Suppose that all the flat connections at the ends of* P *are non-degenerate. Then the operator* $D_\varpi : W^{1,2} \to L^2$ *is a Fredholm operator.*

We explain the additional requirement of acyclicity over the ends needed for the passage from compact manifolds to tubular manifolds with the example of a tube $M = \mathbb{R} \times N$. We have seen that the deformation operator then has the special form $D_\varpi = \frac{d}{dt} - L_{\omega(t)}$, with $L_{\omega(t)}$ a continuous family of self-adjoint operators. Via this relation we can fit it into the general framework of the spectral flow, as derived in A.6, and the statement above is a consequence of proposition A.31, whose proof crucially relies on the fact that the family $L_{\omega(t)}$ converges to invertible operators at $\pm\infty$. For more details see the note after proposition A.31.

Given that the operator D_{ϖ} over an adapted bundle is Fredholm, it admits a well-defined index. In fact, this index is independent of the choice of metric or compatible connection, which allows us to define the **index of an adapted bundle** $\operatorname{ind} P = \operatorname{ind} D_{\varpi}$ using any auxiliary connection ϖ on P. In order to see this, we must distinguish between two different types of perturbations. First, compactly supported perturbations vary the deformation operator continuously through Fredholm operators, thus leaving the index invariant. This follows from the compact theory. Second, perturbations over the tube can be handled by the theory of spectral flow, which gives that as long as the boundary operators do not acquire a kernel during the homotopy of D_{ϖ}, the index will be constant (see A.28). Since $\ker L_{\omega} = \mathrm{H}^0(N, d_{\omega}) \oplus \mathrm{H}^1(N, d_{\omega}) = 0$ is only dependent on the choice of flat limit, which is fixed by the choice of adapted bundle, we see that $L_{\omega(\pm\infty)}$ does indeed not become singular during a perturbation.

Let us now introduce a construction extending the connected sum of two manifolds. Suppose M_0, M_1 are two tubular manifolds, such that M_0 has an end $N \times \mathbb{R}_+$ and M_1 has an end $\bar{N} \times \mathbb{R}_+$, which is oriented diffeomorphic to $N \times \mathbb{R}_-$. We can then glue the two manifolds along this common end given a gluing parameter $T > 0$. First, remove the closed sets $N \times [T, +\infty)$ and $N \times (-\infty, T]$ respectively from M_0 and M_1, and then identify the remaining finite length tubes via the diffeomorphism

$$(x,t) \in N \times (0,T) \mapsto (x,-t) \in N \times (-T,0).$$

We obtain a manifold which has two fewer ends than the disjoint union of M_i, and a neck of length T. Call this manifold $M_\sharp^T = M_0 \sharp_N M_1$. If we have two adapted bundles P_i over M_i, such that the flat structures of the glued ends agree, we can glue them together as well to obtain a family of bundles $P_\sharp^T$. Note that all the manifolds $M_\sharp^T$ and bundles $P_\sharp^T$ are diffeomorphic and only differ by their Riemannian structure through the size of their neck. Therefore, sometimes the superscript T might be dropped if is not essential. If the initial bundles have adapted connections, they agree over the glued ends, so they give rise to an adapted connection over the glued bundle $P_\sharp$. Denote the respective deformation operators by D_0, D_1 and $D_\sharp$. We have the following result relating their indices:

Theorem 4.6 (Infinitesimal gluing) *Let P_0, P_1 be adapted bundles (with only acyclic ends) over tubular manifolds M_0, M_1, which have an end in common. Then:*

$$\operatorname{ind} D_\sharp = \operatorname{ind} D_0 + \operatorname{ind} D_1.$$

Proof The general case can be proven by first treating the case where the operators D_0 and D_1 do not admit a cokernel. Then it can be shown that $D_\sharp$ does not admit a cokernel either, and an explicit isomorphism $\ker D_\sharp \simeq \ker D_0 \oplus \ker D_1$ is obtained through a gluing argument. From there the argument can be adapted to the case where the D_i do have a cokernel by adding operators $K_i : \mathbb{R}^N \to L^2$ such that $D_i \oplus K_i$ is onto and applying the previous argument to them.

We will prove this a bit differently in the case of tubes. In this case, $M_0 = M_1 = \mathbb{R} \times N$, and the gluing conditions translates into gluing the positive end of M_0 with the negative end of M_1. We can then compute the indices of D_i as the spectral flow of the families $L_{\omega_i(t)}$. More precisely we have by proposition A.32:

$$\operatorname{ind} D_i = -\operatorname{sf}(L_{\omega_i(t)}).$$

However the fact that ω_i are adapted over the ends means that the families $(L_{\omega_i(t)})$ are constant away from a compact interval. Homotope the families ω_i so that $L_{\omega_0(t)}$ is constant for $t \geq -1$ and $L_{\omega_1(t)}$ is constant for $t \leq 1$. We see that by lengthening the neck, the family $L_{\omega^\sharp}$ is homotopic to the concatenation:

$$L_{\omega_0(t)}|_{\mathbb{R}_-} \sharp L_{\omega_1(t)}|_{\mathbb{R}_+}.$$

Thus in particular we have by the axioms of the spectral flow:

$$\begin{aligned}
\operatorname{ind} D_\sharp = -\operatorname{sf}(L_{\omega^\sharp}) &= -\operatorname{sf}(L_{\omega_0(t)}|_{\mathbb{R}_-} \sharp L_{\omega_1(t)}|_{\mathbb{R}_+}) \\
&= -\operatorname{sf}(L_{\omega_0(t)}|_{\mathbb{R}_-}) - \operatorname{sf}(L_{\omega_1(t)}|_{\mathbb{R}_+}) \\
&= -\operatorname{sf}(L_{\omega_0(t)}) - \operatorname{sf}(L_{\omega_1(t)}) \\
&= \operatorname{ind} D_0 + \operatorname{ind} D_1.
\end{aligned}$$

□

The result above can be extended to the case where the glued two ends are part of the same connected component, and the additivity then still holds. We can apply this for instance to the case of a tube $M = \mathbb{R} \times N$. Let $[\rho] \in \mathscr{R}(N)$ be a flat acyclic connection, and let P be an adapted bundle where the flat structure on both ends of P is given by some lift of $[\rho]$. Glue the two ends together to obtain a bundle over $S^1 \times N$. By the Künneth formula, and since N is a homology sphere, we have $\mathrm{H}_0(S^1 \times N) \simeq \mathrm{H}_1(S^1 \times N) \simeq \mathbb{Z}$ and $\mathrm{H}_2(S^1 \times N) \simeq 0$. Thus using the formula 4.1 and the gluing relation we obtain for the index:

$$\operatorname{ind} D = \operatorname{ind} D_\sharp = 8c_2(P^\sharp) - 3(1 - 1 + 0) = 8c_2(P^\sharp).$$

Using this we see that there are restrictions on the indices of more general adapted bundles. If $[\rho], [\eta] \in \mathscr{R}(N)$ are two flat connections, and P, P' two adapted bundles having limit $[\rho]$ at $-\infty$ and $[\eta]$ at $+\infty$, we can glue the positive end of P with the orientation reversed positive end of P' to obtain an adapted bundle with flat ends $[\rho]$ on either side. Thus, since reversing the orientation affects the index by negating it, we see:

$$\operatorname{ind} D - \operatorname{ind} \bar{D}' = \operatorname{ind} D^\sharp \in 8\mathbb{Z}, \tag{♥}$$

by the previous result. Thus it makes sense to define the relative index:

$$\delta\left([\eta], [\rho]\right) = \operatorname{ind} D \in \mathbb{Z}_8,$$

where D is the deformation operator associated to any adapted bundle having limits as above. Note that minus the spectral flow between two operators L_η and L_ρ is well-defined in $\mathbb{Z}$ and is a lift of this relative index we have just defined. The problem that arises on the level of gauge equivalence classes is that there are non-trivial loops in $\mathscr{B}^*(N)$, which can change the index by an arbitrary multiple of 8. On the other hand, given two gauge equivalence classes of connections such that $\delta([\eta], [\rho]) = k$, we can construct an adapted bundle of index $8l + k$ for each $l \in \mathbb{Z}$.

The non-acyclic case

We would now like to imitate the finite-dimensional Morse theory, where the relative index $\delta([\eta],[\rho])$ can be refined to the difference of absolute indices $\mu([\eta]) - \mu([\rho])$ of the two critical points. For this we can take a certain flat connection $[\rho] \in \mathscr{R}(N)$ and normalize the index to be zero at it. The indices of other connections then must be given by:

$$\mu([\eta]) = \delta([\eta],[\rho]).$$

This is then well-defined by additivity of the index. But at which connection should we normalise? A natural choice which is distinguished on all homology three-spheres is the trivial connection $[\vartheta]$, which is the unique reducible connection. However so far our definition of δ, or the Fredholm theory in general, does not work on non-acyclic connections such as the trivial one.

We will first show how to extend the Fredholm theory in the case of tubes, and then take a look at how this can be adapted in the general tubular case. If $M = \mathbb{R} \times N$ then we can recast D_ϖ as $\frac{d}{dt} - L_{\omega(t)}$, but where now potentially $L_{\omega(\pm\infty)}$ are singular, and thus D_ϖ may not have closed image as the example of $\frac{d}{dt}$ on $\mathbb{R}$ shows. What can then be done, is that one can perturb the family $L_{\omega(t)}$ such that the limit operators are no longer singular (i.e. the zero eigenspace is pushed up or down to become a slightly positive or negative eigenspace) and then take the spectral flow of that family as the relative index (see section A.6.3 for the details). To be precise we want the kernel to be pushed up at the positive end, and down at the negative end, which in the notation of the appendix is denoted $\mathrm{sf}_{-+}(L_{\omega(t)})$. We define the **Floer index** of a flat connection $[\rho] \in \mathscr{R}(Q)$ to be:

$$\mu([\rho]) = -\,\mathrm{sf}_{-+}(L_\rho, L_\vartheta) \in \mathbb{Z}_8,$$

where ρ is any lift in $\mathscr{C}(Q)$ of $[\rho]$, and ϑ is a trivial connection. For instance if $[\rho] = [\vartheta]$, we can choose as lift $\rho = \vartheta$, so that by proposition A.44 we have $-\,\mathrm{sf}_{-+}(L_\vartheta, L_\vartheta) = -\,\mathrm{sf}_{++}(L_\vartheta, L_\vartheta) - \ker L_\vartheta = 0 - 3 = -3 \equiv 5$. If instead we choose a different trivial connection ρ related to ϑ by a gauge transformation of general degree, we should get different spectral flows, all congruent to 5 mod 8. Indeed, additivity of the non-degenerate spectral leads to:

$$\mathrm{sf}_{-+}(L_\rho, L_\vartheta) + \mathrm{sf}_{+-}(L_\vartheta, L_\vartheta) = \mathrm{sf}_{--}(L_\rho, L_\vartheta) \in 8\mathbb{Z},$$

where sf_{+-} is the perturbed spectral flow where now the negative end gets pushed up and the positive one down. Here the last inclusion follows from the result for acyclic connections. Since by considering sf_{--} we perturb both ρ and ϑ into the same direction, we can in fact perturb them to fixed acyclic connections σ and $\varphi^*\sigma$ respectively. For these we have already in (♥) that $\mathrm{sf}(\sigma, \varphi^*\sigma) \in 8\mathbb{Z}$. Compare this also to example 4.3. If we perturb the start- and end-point of this path in the same way, the boundary points will no longer be on the codimension 1 wall of non-acyclic connections, and the index of the path will be well-defined and equal to $8 \deg \varphi =$

$8 \cdot 1 = 8$. We get that $\mathrm{sf}_{+-}(L_\rho, L_\vartheta) \equiv \mathrm{sf}_{-+}(L_\vartheta, L_\vartheta) \equiv 5 \mod 8$, since in the case of a constant path $A(t)$ we have $\mathrm{sf}_{-+}(A) + \mathrm{sf}_{+-}(A) = \mathrm{sf}_{--}(A) = 0$. By extending this argument to general limits, we see that this new index is indeed well-defined modulo 8. What is left is to see that this is indeed an extension of the index we defined before. If $[\rho], [\eta] \in \mathscr{R}^*(Q)$ are acyclic, we have:

$$\begin{aligned}
\mu([\eta]) - \mu([\rho]) &= -\,\mathrm{sf}_{-+}(L_\eta, L_\vartheta) + \mathrm{sf}_{-+}(L_\rho, L_\vartheta) \\
&= -\,\mathrm{sf}_{-+}(L_\eta, L_\vartheta) - \mathrm{sf}_{+-}(L_\vartheta, L_\rho) \\
&= -\,\mathrm{sf}_{++}(L_\eta, L_\vartheta) - \mathrm{sf}_{++}(L_\vartheta, L_\rho) \\
&= -\,\mathrm{sf}_{++}(L_\eta, L_\rho) = \delta([\eta], [\rho]) \in \mathbb{Z}_8.
\end{aligned}$$

Here we used the concatenation property of the spectral flow, while making sure to only concatenate ends which are perturbed in the same direction (so that the operators can continuously be matched up). Since η and ρ are acyclic, we could use in the last line that $\mathrm{sf}_{++}(L_\eta, L_\rho) = \mathrm{sf}(L_\eta, L_\rho)$, which is a $\mathbb{Z}$-valued lift of $\delta([\eta], [\rho])$. Thus we have completed our goal of defining an absolute index for the critical points, by which one can compute the various indices of the deformation operators of flow lines joining them.

Now, how can this be adapted to the general tubular case? We will not require it in a substantial way, but it is still interesting to see. For a general adapted bundle $P \to M$ over a tubular manifold, choose a weight function $w \in C^\infty(M)$ which over an end $N_i \times \mathbb{R}_+$ is given by $w(p, t) = e^{\alpha_i t}$, where $\alpha_i \neq 0$ is a weight associated to the end N_i. Given a vector $\underline{\alpha} = (\alpha_1, \ldots, \alpha_n)$ we can then define the **weighted Sobolev space** $W^{k,p}_{\underline{\alpha}}(E)$ of sections of a vector bundle $E \to M$ as the closure of the compactly supported smooth sections with respect to the weighted norm:

$$\|s\|_{k,p,\underline{\alpha}} = \|ws\|_{k,p}.$$

For different choices of w with the same weights, the norms are all equivalent, so the Banach structure does only depend on the weights. We can now formulate the Fredholm theory in the non-acyclic case.

Theorem 4.7 ($D^{\underline{\alpha}}_{\varpi}$ is Fredholm) *The operator $D^{\underline{\alpha}}_{\varpi} : W^{1,2}_{\underline{\alpha}} \to L^2_{\underline{\alpha}}$ is a bounded operator between Banach spaces. It is Fredholm if the weight α_i over the end N_i with flat connection ρ_i is not in the spectrum of L_{ρ_i}.*

We can furthermore, in analogy to what we have done before, define the index of an adapted bundle. However in the non-acyclic case this index depends on the weight vector $\underline{\alpha}$. We define thus:

$$\mathrm{ind}(P, \underline{\alpha}) = \mathrm{ind}\, D^{\underline{\alpha}}_{\varpi}.$$

It can be shown that given a weight vector, the index is independent of auxiliary choices, and if the weights are varied so that at all times $\alpha_i \notin \sigma(L_{\rho_i})$, then the index is unchanged. The correct infinitesimal gluing theorem now reads:

Theorem 4.8 (Non-acyclic infinitesimal gluing) *Let P_0, P_1 be adapted bundles over tubular manifolds M_0, M_1. Suppose that M_i have the end $N_1 \times \mathbb{R}_+$ in common, and the weight vectors are of the form $\underline{\alpha}_0 = (\alpha, \underline{\beta}_0)$ and $\underline{\alpha}_1 = (-\alpha, \underline{\beta}_1)$. Then:*

$$\operatorname{ind}\left(P^\sharp, \left(\underline{\beta}_0, \underline{\beta}_1\right)\right) = \operatorname{ind}\left(P_0, \underline{\alpha}_0\right) + \operatorname{ind}\left(P_1, \underline{\alpha}_1\right).$$

All this is best understood in the case of tubes, where the choice of weight function can be expressed as a perturbation of the family $L_{\omega(t)}$. For general tubular manifolds, there are isometries:

$$W^{k,p}_{\underline{\alpha}} \to W^{k,p}; \quad u \mapsto wu.$$

In the case of tubes we can choose $w = e^{\alpha(t)t}$ with $\alpha(t)$ constantly equal to the respective weights α_+, α_- for $|t| >> 0$. In particular we see that under these isometries the operator:

$$\frac{d}{dt} - L_{\omega(t)} : W^{1,2}_{\underline{\alpha}}(\mathbb{R}) \to L^2_{\underline{\alpha}}(\mathbb{R})$$

goes over into the operator mapping $W^{1,2}(\mathbb{R}) \to W^{1,2}_{\underline{\alpha}}(\mathbb{R}) \to L^2_{\underline{\alpha}}(\mathbb{R}) \to L^2(\mathbb{R})$ via the formula:

$$e^{\alpha(t)t}\left(\frac{d}{dt} - L_{\omega(t)}\right)e^{-\alpha(t)t}u = \left(\frac{d}{dt} - \left(L_{\omega(t)} + \alpha(t)\right)\right)u.$$

Thus in other words, defining D_{ϖ} over these weighted spaces has the effect of perturbing the families $L_{\omega(t)}$ over the ends, the same way in which we defined the spectral flow for degenerate ends. In this picture, gluing the positive end of M_0 with the negative end of M_1 is possible exactly when the corresponding weights agree, i.e. $\alpha^0_+ = \alpha^1_-$. The gluing theorem then follows from the concatenation axiom for the spectral flow (since the ends are nicely matched up). One thing to note here is that in the statement of the gluing result, we require the weights over the ends to be opposite of each other. This is because if we re-interpret the negative end of M_1 as a postive end of the form $N \times \mathbb{R}_+$, we have to invert the direction of time, i.e. $t \mapsto -t'$, which means that:

$$e^{\alpha'(t')t'} =' (p, t') = w(p,t) = e^{\alpha(t)t} = e^{-\alpha(t)t'},$$

so that $\alpha'(t') = -\alpha(t)$. Using this theory for tubes with the main theorem A.6.3, the above generalised theorems can be proven by further subdividing the ends, so that the only degenerate gluing happens between honest tube. Then the acyclic results can be used for the tubular manifolds, and the non-acyclic results for the tubes.

4.4 Trajectory spaces

Now that we have set up the Fredholm and index theory over adapted bundles, we can tackle the analysis of instantons on said bundles. For this we first go over the analytic properties of instantons over tubular manifolds, and then run through the necessary compactness and gluing results needed to define the Floer homology groups.

4.4.1 Instantons over tubular manifolds

As the instanton equation is a non-linear elliptic PDE, we have by elliptic bootstrapping that instantons are in fact always smooth, even on the non-compact tubular manifolds. We already indicated in our definition of adapted bundles that instantons converge to a flat connection over each end of a tubular manifold. The precise result is the following:

Proposition 4.9 (Decay of instantons, 4.1 in [1]) *Assume that all flat connections on $Q \to N$ are non-degenerate. Let $\varpi \in \mathscr{C}(P)$ be an instanton such that $\mathrm{F}_\varpi \in L^p$ for some $p \geq 2$. Then for each end, there is a unique flat connection (up to gauge equivalence) $[\rho] \in \mathscr{B}(Q)$ such that $[\omega(t)]$ (corresponding to $[\varpi]$) converges smoothly to $[\rho]$.*

This result, as well as the compactness properties of the moduli space of instantons heavily depends on the following compactness result:

Theorem 4.10 (Uhlenbeck compactness, first version, 2.1 in [1]) *Let $P \to U$ be an $\mathrm{SU}(2)$-bundle over an oriented (not necessarily compact) four-manifold and $\varpi_k \in \mathscr{C}(P)$ a sequence of instantons, such that $\|\mathrm{F}_{\varpi_k}\|_{L^2} \leq C < 8\pi^2$. Then there is a subsequence that converges smoothly on compact subsets in $\mathscr{B}(P)$.*

In this statement we included the condition $\mathrm{F}_{\varpi_k} \leq C < 8\pi^2$ in order to prevent **bubbling**, meaning the concentration and disappearance of curvature in the limit.

Proof (of proposition 4.9) Consider an end $\mathbb{R}_+ \times N$, and partition it into bands $B_k = (k, k+1) \times N$. After a translation, consider the sequence of instantons $\varpi_k = \varpi|_{B_k}$ as being defined on $(0,1) \times N$. By Hölder's inequality, the L^p-bound on the curvature means that $\|\varpi_k\|_2 \to 0$ for $k \to \infty$. Thus by Uhlenbeck compactness, we can find a convergent subsequence, also denoted by ϖ_k, converging smoothly on compact sets to a flat connection $[\rho] \in \mathscr{B}(P|_{B_0})$. Since flat connections on B_0 are paths of flat connections on N, and since the flat connections are non-degenerate, and hence isolated, we have that $[\omega(k+\frac{1}{2})] \to [\rho(\frac{1}{2})]$ in $\mathscr{B}(Q)$. Finally, the path $[\omega(t)]$ is continuous and converges to $\mathscr{R}(Q)$ in L^2 by the curvature bound. Since the set of flat connections is a discrete subset of the Hausdorff space $\mathscr{B}(Q)$, the limit $\lim_{t\to\infty}[\omega(t)]$ exists and is equal to $[\rho(\frac{1}{2})]$. $\square$

This convergence result means that if all flat connections on N are non-degenerate, we can associate to every instanton an adapted bundle, for which it will then be a compatible connection. The analogous result in Morse theory for compact manifolds is that every flow line has a determined start and end point. We can say even more about this decay.

Proposition 4.11 (Exponential decay) *Let $\varpi \in \mathscr{C}(P)$ be an instanton with non-degenerate limiting connections $\{\rho_i\}_{1\leq i\leq n}$. Let $\delta > 0$ be the smallest positive eigenvalue of $H_{\rho_i}|_{\delta_{\rho_i}}$, and suppose that $\mathrm{F}_\varpi \in L^p$ for some $p \geq 2$. Then :*

$$\|\mathrm{F}_{\omega(t)}\| \leq Ce^{-\delta t}.$$

The proof relies on establishing a differential inequality for $J(t) = \int_t^\infty \|F_{\omega(t)}\|^2$ of the form $\frac{dJ}{dt} \leq \delta J^2 + CJ^3$. Then there is the following result concerning the connection potential of an instanton over an end:

Proposition 4.12 (Exponentially decaying gauges) *Let $\varpi \in \mathscr{C}(P)$ be an instanton with non-degenerate limiting connections $\{\rho_i\}_{1\leq i\leq n}$. Let $\delta > 0$ be the smallest positive eigenvalue of $H_{\rho_i}|_{\delta_{\rho_i}}$, and suppose that $F_\varpi \in L^p$ for some $p \geq 2$. Then over the i-th end we can represent the connection as $\varpi = \rho_i + a$ where $|\nabla^l a| \in O(e^{-\delta t})$.*

Since the base manifold M is non-compact, we need to be careful about choosing a Sobolev space of connections. A good choice needs to take into account the adapted structure of the bundle P. We choose $\mathscr{C}^{1,p}(P) = \varpi_0 + W^{1,p}(M, \mathrm{ad}\, P)$, where ϖ_0 is an adapted connection. In other words it is the completion of the space of connections with flat ends and a given second Chern number. We denote the set of instantons on P by $\mathscr{M}_P$. As we have already stated, the map $F^+ : \mathscr{C}^{1,p}(P) \to L^p(\Omega^{2+}(M, \mathrm{ad}\, P))$ is a smooth map and the **moduli space of instantons** $\mathscr{M}_P = (F^+)^{-1}(0)$ is the pre-image of the zero-section. We say that a connection $\varpi \in \mathscr{C}^{1,p}(P)$ is a regular point if the differential d_ϖ^+ (and thus also the operator D_ϖ) is surjective, and admits a bounded right inverse. In this case the dimension of $\ker D_\varpi$ is given by $\mathrm{ind}\, D_\varpi$. Here is the precise statement for the moduli space of instantons.

Proposition 4.13 (Moduli space of instantons, 4.13 in [1]) *Let M be tubular, and P be an adapted bundle with acyclic ends. For generic smooth metrics, all non-flat instantons are regular, and all moduli spaces $\mathscr{M}_P$ with $c_2(P) \neq 0$ are smooth manifolds of dimension* $\dim \mathscr{M}_P = \mathrm{ind}\, P$.

This is a consequence of the inverse function theorem for Banach spaces, as well as the Sard-Smale therem, the infinite dimensional analogue of Sard's lemma, which states that for a map between finite dimensional manifolds $f : X \to Y$, the set of regular values is open dense. In fact we can achieve regularity of the moduli space by perturbing the metric in a compact set, thus we do not need to vary the metrics over the ends.

4.4.2 Gluing

Suppose we have two adapted bundles $P_i \to M_i$ $(i = 0, 1)$ over tubular manifolds, which have an end $N \times \mathbb{R}_+$ in common. Glue them together over this end to obtain a bundle $P_\sharp^T \to M_0 \sharp_N M_1$. It turns out that under suitable circumstances we can extend the infinitesimal gluing 4.6 to a non-linear **gluing map** of instantons of the form

$$\tau_T : \mathscr{M}_{P_1} \times \mathscr{M}_{P_2} \hookrightarrow \mathscr{M}_{P^\sharp}.$$

Suppose we have regular instantons $\varpi_i \in \mathscr{M}_{P_i}$. By the decay results along tubes 4.9 and 4.12 we know that if we consider these instantons on $N \times [T-2, \infty)$ for T very large, both the curvature and the local gauge potential will be of magnitude $O(e^{-\lambda T})$. Using partitions of unity, we can deform the instantons $\varpi_i = \rho + a_i$ over $[T-2, \infty)$

to connections $\tilde{\varpi}_i$ which are flat on $[T-1,\infty)$, i.e. equal to ρ. Thus the self-dual part of the curvature of $\tilde{\varpi}_i$ is supported in $[T-2, T-1]$. This now allows us to glue the two new connections together to obtain a connection $\varpi_0\sharp\varpi_1 \in \mathscr{C}(P_\sharp)$, which has self-dual curvature away from a finite-length strip, and agrees with ϖ_i if restricted to the complement of $N \times [T-2,\infty)$ in M_i. Moreover the self-dual curvature is of magnitude $O(e^{-\lambda T})$ by the decay result, hence the glued connection is in a sense *almost an instanton*. To obtain the true instanton $\tau_T(\varpi_0, \varpi_1) \in \mathscr{M}_{P_\sharp}$ the idea is now to find the instanton that is *closest* to our initial guess $\varpi_0\sharp\varpi_1 = \tilde{\varpi}_\sharp$. Consider therefore a perturbation $\varpi^\sharp = \tilde{\varpi}_\sharp + a$ with a small. The instanton equation for $\varpi^\sharp$ can be written as:

$$d^+_{\tilde{\varpi}_\sharp} a + (a \wedge a)^+ = -\mathrm{F}_{\tilde{\varpi}_\sharp}.$$

Note that from the direct proof of the infinitesimal gluing given in [1] it can be seen that since ϖ_i were regular instantons, i.e. have $d^+_{\varpi_i}$ surjective, the glued operator $d^+_{\tilde{\varpi}_\sharp}$ is surjective as well, and thus has a bounded right-inverse Q. We try to find a solution of the form $a = Q(\varphi)$, i.e. to solve the equation:

$$\varphi + (Q(\varphi) \wedge Q(\varphi))^+ = -\mathrm{F}_{\tilde{\varpi}_\sharp}.$$

This can be done using the Banach fixed point theorem. Note that solutions to the above equation are exactly the fixed points of the map:

$$\varphi \mapsto T(\varphi) = -\mathrm{F}_{\tilde{\varpi}_\sharp} - (Q(\varphi) \wedge Q(\varphi))^+.$$

Now the trick is to define the right function spaces, so that the above becomes a contraction, in the sense that $\|T(\varphi) - T(\psi)\| \leq C\|\varphi - \psi\|^2$. Thus for small enough $\|\mathrm{F}_{\tilde{\varpi}_\sharp}\|$, the Banach fixed point theorem yields the existence of exactly one fixed point in proximity of $\varphi_0 = \mathrm{F}_{\tilde{\varpi}_\sharp}$. Thus uniqeness is also guaranteed. The details are worked out in ([1], p.94f). If we now restrict to pre-compact subsets $N_i \subset \mathscr{M}_{P_i}$ of regular instantons, we can perform the above construction uniformly to yield a map:

$$\tau_T : N_1 \times N_2 \to \mathscr{M}_{P\sharp}.$$

Note that the bigger T is, the longer the resulting instanton spends in a neighbourhood of the flat connection ρ on the connecting tube. Comparing with the classical Morse theory picture, if $T \to +\infty$ then the flow lines $\tau_T(\varpi_1, \varpi_2)$ will break along the flat connection ρ into its constituent parts ϖ_i.

If the glued instanton $\tau_T(\varpi_0, \varpi_1)$ is regular as well, we see from the infinitesimal gluing that the dimensions of the moduli spaces add up:

$$\dim \mathscr{M}_{P_0} + \dim \mathscr{M}_{P_1} = \dim \mathscr{M}_{P_\sharp}.$$

Thus it is reasonable to expect that by varying ϖ_i we can obtain all instantons in a neighbourhood of $\tau_T(\varpi_0, \varpi_1)$, i.e. that the gluing map is a local diffeomorphism. Furthermore, since we want to use the gluing construction in the end to compactify the spaces, we need to be able to reach via τ_T all the connections that escape towards the boundary, so that we know which broken flow lines they converge to. Here the precise notion of closeness in $\mathscr{M}_{P_\sharp}$ is given by the metric induced from the L^2-norm on $\mathscr{C}(P)$, and we can relate instantons $\varpi_\sharp \in \mathscr{M}_{P_\sharp}$ with potential pieces $\varpi_i \in \mathscr{M}_{P_i}$ via the comparison distance:

$$d_{comp}(\tau_T(\varpi_0, \varpi_1); \varpi_0, \varpi_1) = d_{L^2}([\varpi_\sharp|_{G_0}], [\varpi_0|_{G_0}]) + d_{L^2}([\varpi_\sharp|_{G_1}], [\varpi_1|_{G_1}].)$$

Here $G_i \subset M_i$ are compact subsets of M_i. If the G_i are big enough (i.e. they are the complement in M_i of the ray $N \times (T-2, \infty]$ for instance), and T is big enough, then this metric captures very well the notion of a connection in $\mathscr{M}_{P_\sharp}$ being a concatenation of connections in $\mathscr{M}_{P_i}$, as the following result, the main theorem about gluing, shows. We will state it here assuming that all limits are acyclic, since that is the only case we require.

Theorem 4.14 (Gluing of instantons, 4.17 in [1]) *Let S_i be compact sets of regular points in $\mathscr{M}_{P_i}$. For small enough $\kappa > 0$ and for large enough $T > 0$, there are neighborhoods $S_i \subset V_i$ and a smooth map:*

$$\tau_T : V_0 \times V_1 \to \mathscr{M}_{P^\sharp},$$

such that:

- *τ_T is a diffeomorphism onto its image.*
- *$d_{comp}(\tau_T(\varpi_0, \varpi_1); \varpi_0, \varpi_1) \leq \kappa$ for all $\varpi_i \in V_i$.*
- *Any connection $\varpi^\sharp \in \mathscr{M}_{P^\sharp}$ with $d_{comp}(\varpi^\sharp; \varpi_0, \varpi_1) \leq \kappa$ for some $\varpi_i \in N_i$ lies in the image $\tau_T(V_0 \times V_1)$.*

Note that if the moduli spaces $\mathscr{M}_{P_i}$ are themselves regular and compact we get a gluing map of the form:

$$\tau_T : \mathscr{M}_{P_1} \times \mathscr{M}_{P_2} \hookrightarrow \mathscr{M}_{P^\sharp}.$$

4.4.3 Compactness

Let $P \to M$ be an adapted bundle. We would like to investigate the compactness properties of the moduli space of instantons on this bundle $\mathscr{M}_P$. We have the following result by Uhlenbeck, of which a weaker version was already introduced before in 4.10.

Theorem 4.15 (Uhlenbeck compactness, 2.1 in [1]) *Let $P \to U$ be an $\mathrm{SU}(2)$-bundle over an oriented (not necessarily compact) four-manifold and $\varpi_k \in \mathscr{C}(P)$ a sequence of instantons, such that $\|\mathrm{F}_{\varpi_k}\|_{L^2} \leq C$. Then there is a subsequence (which we also denote by ϖ_k) that converges in the following weak sense. There are a finite number of points $x_1, \ldots, x_l \in U$, with $l \leq \frac{C}{8\pi^2}$ and an instanton $\varpi \in \mathscr{C}(P|_{U \setminus \{x_1, \ldots, x_l\}})$, such that:*

$$[\varpi_k] \to [\varpi] \textit{ in } \mathscr{B}(P|_{U \setminus \{x_1, \ldots, x_l\}}),$$

and the curvature densities $|F_{\varpi_k}|^2$ *converge to the measure:*

$$|F_{\varpi_k}|^2 + 8\pi^2 \sum_{i=1}^{l} \delta_{x_i}.$$

We will now use this basic compactness result to derive the compactness up to broken trajectories of the moduli spaces of CS-flowlines. But first, let us give an indication of what Uhlenbeck compactness means in our situation. Suppose therefore that U is a tube. Then there are two ways in which strong compactness can fail. First, there is the possibility that curvature can concentrate at a point and vanish in the limit. Each time that happens the norm of the curvature of the limit is reduced by exactly $8\pi^2$. This is the so called **bubbling** phenomenon, which essentially arises because the basic instanton on $\mathbb{R}^4$ can be shrunk arbitrarily. More formally, the set of instantons on S^4 admits a free group action of the non-compact group $\mathbb{R}_{>0}$ via scaling, and by a map $U \to S^4$ this disappearing family of instantons can be grafted onto any manifold. Then there is the sliding-off of curvature of a tube, via the action of $\mathbb{R}$ on the instantons over the tube by translation. As an example, let $c_T : N \times \mathbb{R} \to N \times \mathbb{R}$ denote the translation $(p, t) \mapsto (p, t + T)$, then for any instanton $\varpi \in \mathscr{C}(P)$, if $\{T_k\}_{k \in \mathbb{N}}$ is a sequence diverging to $+\infty$, then $c_{T_i}^* \varpi$ converges weakly to a flat instanton. Note that if we think of $S^3 \times \mathbb{R}$ as the conformally equivalent space $\mathbb{R}^4 \setminus \{0\}$, then sliding off to infinity is simply bubbling at $0 \in \mathbb{R}^4$. To counteract this second kind of loss of curvature, we introduce the **reduced moduli space** $\mathscr{M}_P' \subset \mathscr{M}_P$, which consists of all the **centred** instantons, i.e. instantons $\varpi \in \mathscr{C}(P)$ such that:

$$\int_{-\infty}^{+\infty} t|F_{\varpi}|^2 = 0.$$

If the instanton ϖ is not flat, then there is exactly one centred translate of it. Hence if $\mathscr{M}_P$ only consists of non-flat instantons (which is the case if $c_2(P) \neq 0$), then $\mathscr{M}_P' \times \mathbb{R} = \mathscr{M}_P$. On the other hand, if $c_2(P) = 0$, then all instantons are flat, and as such $\mathscr{M}_P' = \mathscr{M}_P$.

We now introduce the notions of weak and strong convergence on the moduli space of instantons on an adapted bundle $P \to N \times \mathbb{R}$, for which we have compactness results. A **translation vector** $\underline{T}$ is an ordered sequence $T(1) < T(2) < \cdots < T(n)$ of real numbers. Let $[\varpi_\alpha] \in \mathscr{M}_P$ be a sequence of instantons and $[\varpi_1] \in \mathscr{M}_{P_1}, \ldots, [\varpi_n] \in \mathscr{M}_{P_n}$ a collection of instantons on possibly different bundles P_i. We say that $[\varpi_\alpha]$ is $\underline{T}_\alpha$**-chain-convergent** to $[\underline{\varpi}] = ([\varpi_1], \ldots, [\varpi_n])$ if there is a sequence of translation vectors $\underline{T}_\alpha$ with $\underline{T}_\alpha(k) - \underline{T}_\alpha(k-1) \to +\infty$, such that:

$$\left[c^*_{\underline{T}_\alpha(k)}\right] \to [\varpi_k]. \tag{†}$$

We say that the chain-convergence is weak if (†) holds only in the weak sense from Uhlenbeck's theorem. Define a **chain** of adapted bundles subdividing a bundle P to be a collection $\underline{P} = (P(1), \dots, P(n))$ of adapted bundles with the following property. If P has flat limits ρ^- and ρ^+, and P_i have flat limits ρ_i^- and ρ_i^+ respectively, then we must have:

$$\rho^- = \rho_1^-, \quad \rho_1^+ = \rho_2^-, \quad \dots, \quad \rho_{n-1}^+ = \rho_n^-, \quad \rho_n^+ = \rho^-.$$

A **weak chain** is a chain $\underline{P}$ that also encodes at which points on an intermediate tube bubbling occurs. To be precise this additional data is a collection $\underline{Z} = (Z(1), \dots, Z(n))$, where $Z(i)$ is a finite, unordered collection of points on $N \times \mathbb{R}$. A chain is also always a weak chain by setting $Z(i) = \emptyset$. In the following we will therefore refer to both chains and weak chains simply as chains. For $(\underline{P}, \underline{Z})$ be a chain of adapted bundles, we define its index as:

$$\operatorname{ind}(\underline{P}, \underline{Z}) = \sum_i (\operatorname{ind} P_i + 8|Z(i)|).$$

Furthermore, we define the second Chern number of a chain as:

$$c_2(\underline{P}, \underline{Z}) = \sum_i (c_2(P_i) + |Z(i)|).$$

Thus in the language of chains, a sequence of instantons $[\varpi_\alpha] \in \mathscr{M}_P$ weakly chain-converges, if there is a chain $(\underline{P}, \underline{Z})$ and a collection of instantons $[\underline{\varpi}]$ with $\underline{\varpi}_i \in \mathscr{M}_{P(i)}$, such that correctly chosen translates of $[\varpi_\alpha]$ converge to $[\varpi_i]$ locally in C^∞ away from the bubbling points $Z(i)$. The convergence is strong if $|Z(i)| = 0$. Note that we can always introduce intermediate flat bundles into the chain by choosing the translates correctly. For instance if we choose T_α such that $\int_{-\infty}^{T_\alpha} |\mathrm{F}_{\varpi_\alpha}|^2 < \frac{1}{\alpha}$, the translates $c_{T_\alpha}^* \varpi_\alpha$ will converge to a flat instanton, which in $\mathscr{B}(Q)$ just is a constant path. Thus if we speak of the length of a chain, we mean the shortest possible length. The main compactness theorem, on which all of the subsequent analysis of the trajectory spaces is based is the following:

Theorem 4.16 (Weak Compactness) *Any sequence of instantons over an adapted bundle $P \to N \times \mathbb{R}$ has a weakly chain-convergent subsequence. Let a limit chain be $(\underline{P}, \underline{Z})$. There always holds $c_2(\underline{P}, \underline{Z}) \leq c_2(P)$. If in addition $c_2(\underline{P}, \underline{Z}) = c_2(P)$, then the convergence is strong.*

There is a corresponding theorem for the general case of tubular manifolds, where the notion of chain convergence has of course to be adapted. It needs to include an adapted bundle over the original tubular manifolds, as well as tubular chains as we have introduced them above over each end. In particular, using this extended theorem, the above result does not only hold in the case where $M = N \times \mathbb{R}$ with the product metric, but also when the the metric varies with t, i.e. when regarding the tube as a tubular manifold with two ends, and a center piece diffeomorphic to $N \times (-1, 1)$. In order to constrain the possible limit connections of a sequence more, we have the following result about the index of the limit chain:

Proposition 4.17 (Index of limit, 5.7 in [1]) *Let $[\varpi_\alpha] \in \mathscr{M}_P$ be a sequence that is weakly chain-convergent to $[\underline{\varpi}_\alpha]$ on $(\underline{P}, \underline{Z})$. Then we have:*

$$\operatorname{ind}(P) = \operatorname{ind}(\underline{P}, \underline{Z}) + H.$$

Here H denotes the sum of $\mathrm{H}^0(N, d_{\rho_i})$ for all the intermediate flat connections ρ_i.

From these results we can now prove the main compactness principles for low-dimensional trajectory spaces:

Corollary 4.18 (Compactness principle) *Suppose all moduli spaces over the tube $N \times \mathbb{R}$ are regular. We then have the following for an adapted bundle P over the tube:*

- *If $\dim \mathscr{M}_P < 9$, then any sequence has a* ***strongly*** *chain-convergent subsequence.*
- *If $\dim \mathscr{M}_P < 5$, then the chain has length $\leq \dim \mathscr{M}_P + 1$ and does not factor through trivial limits. If $N \times \mathbb{R}$ is a Riemannian tube then the length is $\leq \dim \mathscr{M}_P$.*

Proof

- Let $(\underline{P}, \underline{Z})$ be the limit chain. Then by proposition 4.17 we have:

$$\begin{aligned} 9 > \operatorname{ind}(P) = \operatorname{ind}(\underline{P}, \underline{Z}) + H = & \sum_i (\operatorname{ind} P_i + 8|Z(i)|) + H \\ \geq & 8 \sum_i |Z(i)| + 1 > 0. \end{aligned}$$

  As such $\sum_i |Z(i)|$ must be zero, so no bubbling can occur. Here we used the fact that since all the moduli spaces are regular and non-empty, the must have positive index. Furthermore, by translation invariance, the index of every element in the chain must in fact be ≥ 1. Since the limit chain must at least contain one term, we thus have $\sum_i \operatorname{ind} P_i + H \geq 1 + 0 = 1$.

- Recall that $\dim \mathrm{H}^0(N, d_\vartheta) = 3$ for the three-dimensional trivial connection. Thus if the chain were to factor through the trivial connection, again by regularity all the indices of adapted bundles over true Riemannian tubes $(N \times \mathbb{R}, g^2 + dt^2)$ must be ≥ 1, so that in particular, the two terms on either end must have index ≥ 1. Note that there might be an intermediate term were the metric is varied, so that the translation invariance cannot be applied. This intermediate bundle might have index 0. In all we get that:

$$5 > \operatorname{ind}(P) = \operatorname{ind}(\underline{P}, \underline{Z}) + H \geq 1 + 3 + 1 = 5,$$

  describing the situation where the limit chain has 2 terms (the minimum needed to factor through any connection) and the intermediate connection is trivial. Thus the chain cannot factor through acyclic limits. Since every intermediate bundle except for potentially the one where the metric is non-constant has index at least 1, the result on the chain length follows. □

The gluing construction on tubes we introduced last section can intuitively be seen as an inverse to the splitting of adapted bundles chains when taking limits. Suppose we are in the setting of theorem 4.14, where $S_i \subset \mathscr{M}_{P_i}$ is a pre-compact set of instantons. Then we have the following result.

Proposition 4.19 (Relation to gluing, 5.8 in [1]) *If $[\varpi_i] \in S_i$ are regular instantons and $[\varpi_\alpha]$ is a sequence of centred connections over $N \times \mathbb{R}$ which is strongly chain-convergent to $([\varpi_0], [\varpi_1])$ on $\underline{P} = (P_0, P_1)$, then for large enough T, the connections $[\varpi_\alpha]$ lie in the image of the gluing map τ_T.*

Combining the previous two results, we can now describe more precisely the compactness behaviour of 0, 1 and 2−dimensional trajectory spaces on a homology three-sphere.

Theorem 4.20 (Compactness up to broken trajectories) *Let N be a homology three-sphere. Let $N \times \mathbb{R}$ be a tube with a (not necessarily constant metric) such that all trajectory spaces $\mathscr{M}_P$ are regular. Let P be a fixed adapted bundle joining two acyclic connections. We then have the following compactness properties:*

- *If $\dim \mathscr{M}_P = 0$, then $\mathscr{M}_P$ is compact.*
- *If $\dim \mathscr{M}_P = 1$ and let the metric on the tube be constant, then the reduced moduli space $\mathscr{M}'_P$ is compact.*
- *If $\dim \mathscr{M}_P = 2$ and let the metric on the tube be constant, then any sequence of centred instantons in $\mathscr{M}'_P$ either has a convergent subsequence, converging to a limit in $\mathscr{M}'_P$, or a subsequence which is strongly chain convergent $[\underline{\varpi}]$ on a two-step chain to $\underline{P} = (P(1), P(2))$, where the $P(i)$ have acyclic limits. Thus we can compactify $\mathscr{M}'_P$ to a 1-dimensional manifold by adding in the broken flow lines $\mathscr{M}'_{P(1)} \times \mathscr{M}'_{P(2)}$.*

Proof

- Any sequence in $\mathscr{M}_P$ must contain a strongly chain-convergent subsequence by the compactness principle 4.18. Since the chain however cannot be subdivided further in a meaningful way (that is without appending chains that connect a connection to itself), the convergence must be convergence in $\mathscr{M}_P$.
- Again, by the compactness principle, any sequence must contain a strongly convergent subsequence of length at most 1. Thus is is in fact convergence up to translation in the same moduli space. Choosing the normalisation by centring all instantons thus gives convergence of centred instantons towards a centred instanon on P.
- Here the limit chain can either have length one, so that by the previous argument the convergence is strong in $\mathscr{M}'_P$, or it can have length 2. Since the limit cannot factor through the trivial connection (which on homology three-spheres is the only reducible connection) it must factor through irreducible connections.

Via the gluing map and proposition 4.19, which tells us that any such sequence of divergent instantons can be obtained by gluing elements in $\mathscr{M}'_{P(i)}$, we see that by adjoining the broken flow lines, we can compactify $\mathscr{M}'_P$. □

4.4.4 Perturbation

It may happen that not all critical points of CS are non-degenerate. We then need to perturb the functional to achieve non-degeneracy. This must however be done carefully, since the resulting Floer groups should not depend on the perturbation. We will for this purpose introduce a special class of perturbations adapted for the case of homology three-spheres. Let thus $Q \to N$ be a trivial $\mathrm{SU}(2)$-bundle over a three-manifold, and choose a knot (i.e. an embedding of a circle) $\gamma : S^1 \to N$. Its normal bundle $N\gamma \to S^1$ is an orientable bundle over S^1, and thus trivial. By the tubular neighbourhood theorem we can therefore thicken the loop to an embedding of a solid torus $\Gamma : S^1 \times \mathbb{D}^2 \to N$. For $z \in \mathbb{D}^2$, define the longitude $\gamma_z : S^1 \to N$ to be $\gamma_z(t) = \Gamma(t, z)$. Finally, choose a compactly supported two-form $\mu \in \Omega^2(\mathbb{D}^2)$ with integral 1. We define for $\omega \in \mathscr{C}(Q)$:

$$\sigma(\omega) = \int_{\mathbb{D}^2} \mathrm{tr}\left(\hat{\mathbb{P}}^{\omega}_{\gamma_z}\right)\mu.$$

Here $\hat{\mathbb{P}}^{\omega}_{\gamma_z}$ denotes the holonomy of ω around the loop γ_z, which is an element of $\mathrm{SU}(2)$. Since the holonomies of gauge-equivalent connections are conjugated, this is a gauge-invariant mapping. Moreover it can be shown that σ is a C^2-map on $\mathscr{B}^*(Q)$. Choosing a collection of such knots γ_i for $1 \leq i \leq K$ we can in this manner obtain functionals σ_i. For a vector $\underline{\varepsilon} \in \mathbb{R}^K$ we can define the functional:

$$\underline{\sigma}^{\underline{\varepsilon}} : \mathscr{C}(Q) \to \mathbb{R}; \quad \omega \mapsto \sum_{i=1}^{K} \varepsilon_i \sigma_i(\omega),$$

which of course also descends to define a functional $\underline{\sigma}$ on $\mathscr{B}^*(Q)$. We call functions of this type **admissible perturbations**. The main result on these perturbations is the following:

Theorem 4.21 *If K is sufficiently large, then there are loops γ_i for $1 \leq i \leq K$ such that the functional* $\mathrm{CS} + \underline{\sigma}^{\underline{\varepsilon}}$ *only has non-degenerate critical points on $\mathscr{C}(Q)$, where $\underline{\varepsilon}$ can be chosen arbitrarily small in an open dense neighbourhood of* $0 \in \mathbb{R}^K$*. Moreover the perturbation can be chosen such that for a given metric on N, all moduli spaces on the tube $N \times \mathbb{R}$ are regular.*

Note that if N is a homology three-sphere, then the trivial connection is already non-degenerate for the usual Chern-Simons functional. Thus sufficiently small perturbations keep it non-degenerate.

4.5 Floer homology groups

Consider a homology three-sphere N. Then the set of critical points $\mathscr{R}^*(Q)$ of CS is finite. Perturb the Chern-Simons functional so that every critical point is non-degenerate, and choose a metric on N such that all the moduli spaces of instantons over $N \times \mathbb{R}$ are regular. For simplicity, we will be working with the Chern-Simons functional directly and assume that all the critical points are already non-degenerate. We can now define the chain complex which underlies instanton Floer homology.

Definition 4.22 (Floer chain complex) *Let g be a Riemannian metric on N such that all moduli spaces M_P for adapted bundles over $N \times \mathbb{R}$ are regular. We then define the **Floer chain complex** of the pair (N, g) to be the $\mathbb{Z}_8$-graded chain complex which in degree $k \in \mathbb{Z}_8$ is given by the $\mathbb{Z}_2$-vector space*

$$\mathrm{CF}^k(N) = \mathbb{Z}_2\langle[\rho] \in \mathscr{R}^*(Q) : \mu([\rho]) = k\rangle.$$

The differential $d_g : \mathrm{CF}^k(N) \to \mathrm{CF}^{k-1}(N)$ is given by:

$$d_g[\rho] = \sum_{\mu(\eta)=k-1} \langle \rho, \eta \rangle [\eta].$$

Here $\langle \rho, \eta \rangle := |\mathscr{M}'_P| \mod 2$, where P is an adapted bundle of index 1 joining $\rho \to \eta$.

Proposition 4.23 *The Floer chain complex is a chain complex, i.e. $d^2 = 0$.*

Proof Direct computation of d_g^2 gives for $[\rho] \in \mathrm{CF}^k(N)$:

$$d_g^2[\rho] = d_g \sum_{\mu(\eta)=k-1} \langle \rho, \eta \rangle [\eta] = \sum_{\mu(\tau)=k-2} \sum_{\mu(\eta)=k-1} \langle \rho, \eta \rangle \langle \eta, \tau \rangle [\tau].$$

So we need to show that for any $[\tau] \in \mathrm{CF}^{k-2}(N)$ we have $\sum_{\mu(\eta)=k-1} \langle \rho, \eta \rangle \langle \eta, \tau \rangle = 0$. But by the theorem 4.20 on compactness up to broken trajectories we know that this sum is exactly the number of boundary points of the compact manifold $\mathscr{M}'_P$, where P is the bundle having limits ρ and τ respectively. By the classification of compact one-manifolds it is always even. □

Thus the **Floer homology groups**:

$$\mathrm{HF}^k(N, g) := \mathrm{H}^k(\mathrm{CF}(N), d_g)$$

are well defined $\mathbb{Z}_2$-vector spaces, which are a priori dependent on the metric g.

4.5.1 Functoriality

A priori, the Floer homology group as we have defined them could depend on the choice of metric, which enters substantially into the definition of the moduli space of instantons. Furthermore it needs to be checked that all admissible perturbation of CS give rise to isomorphic homology groups for the theory to be useful.

Independence of the metric

Let g_0, g_1 be two metrics on a homology three-sphere N, such that the Floer homology groups $\mathrm{HF}^*(N, g_i)$ are well-defined. We will define a map associated to these two metrics:

$$\mathrm{HF}(g_0, g_1) : \mathrm{HF}^*(N, g_0) \mapsto \mathrm{HF}^*(N, g_1),$$

such that if g_2 is another metric with associated well-defined Floer homology groups, the following functorial property is satisfied:

$$\mathrm{HF}(g_1, g_2) \circ \mathrm{HF}(g_0, g_1) = \mathrm{HF}(g_0, g_2),$$

and $\mathrm{HF}(g_0, g_0) = \mathrm{id}$. We thus see that the Floer homology groups define a functor from the groupoid category with objects the regular Riemannian metrics on N, and exactly one morphism between each ordered pair of metrics. Since functors preserve isomorphisms, every map $\mathrm{HF}(g_0, g_1)$ must be an isomorphism. Now onto how these maps are defined. We proceed along the following program:

1. Define the map $\mathrm{CF}(g_1, g_2)$ on the chain level using an auxiliary metric G on the tube $N \times \mathbb{R}$.

2. Prove that this map is a chain map, and as such descends to homology.

3. Prove that while the choice of metric G may impact the map $\mathrm{CF}(g_1, g_2)$, any two such maps are chain-homotopic, thus descend to the same map in homology.

4. Prove the functorial properties of the chain map, which imply the functorial properties of the map in homology.

We now go through the steps in more detail.

(1) Let $\rho, \eta \in \mathrm{R}^*(N)$ be two flat connections such that $\mu(\rho) = \mu(\eta)$. Thus there is an adapted bundle $P(\rho, \eta)$ over the tube $N \times \mathbb{R}$ of index 0 and with limits ρ as $t \to -\infty$ and η as $t \to +\infty$. Choose a metric G on the tube, which agrees with g_0 on $N \times (-\infty, -1]$ and g_1 on $N \times [1, +\infty)$, such that all the moduli spaces are regular. Then as a consequence of theorem 4.20 we have that $\mathscr{M}_{P(\rho,\eta)}$ is a compact 0-manifold, i.e. a finite set. We define the map:

$$\mathrm{CF}_G(g_0, g_1)([\rho]) = \sum_{\mu([\eta])=\mu([\rho])} |\mathscr{M}_{P(\rho,\eta)}|[\eta].$$

This assignment is well-defined, but a priori depends on the choice of metric G.

(2) To prove that $\mathrm{CF}_G(g_0, g_1)$ is a chain map, we need to show that the map:

$$F = d \circ \mathrm{CF}_G(g_0, g_1) - \mathrm{CF}_G(g_0, g_1) \circ d : \mathrm{CF}^k(N) \to \mathrm{CF}^{k-1}(N)$$

vanishes identically. We consider the coefficient relating $[\rho] \in \mathrm{CF}^k(N)$ to $[\tau] \in \mathrm{CF}^{k-1}(N)$, which is given by:

$$\sum_{\mu(\eta)=k} |\mathscr{M}_{P(\rho,\eta)}|\langle \eta, \tau\rangle - \sum_{\mu(\eta)=k-1} \langle \rho, \eta\rangle|\mathscr{M}_{P(\eta,\tau)}|.$$

To show that this coefficient vanishes, we note that it counts the boundary points of the compact one-manifold $\mathscr{M}_{P(\rho,\tau)}$. Indeed a sequence of instantons over this bundle of index one with varying metric can break into at most two flow lines (since all bundles are assumed regular), one of index 0 (where the metric changes, and thus admits no translation invariance) and one of index 1 (on either end, where the metric is constant).

3 We can show independence of the metric similarly. Indeed for two auxiliary metrics G_0 and G_1, as well as a homotopy G_t between them, we construct a chain homotopy between $\mathrm{CF}_{G_0}(g_0, g_1)$ and $\mathrm{CF}_{G_1}(g_0, g_1)$, i.e. a map:

$$H : \mathrm{CF}^k(N, g_0) \to \mathrm{CF}^{k+1}(N, g_1),$$

such that $dH + Hd = \mathrm{CF}_{G_0}(g_0, g_1) - \mathrm{CF}_{G_1}(g_0, g_1)$. Its existence shows that the two chain maps defined using different auxiliary data descend to define the same map in homology. The chain homotopy H is defined through its matrix representation, namely by counting pairs (ϖ, t) such that there is an instanton ϖ with respect to the metric G_t joining $[\rho] \in \mathrm{CF}^k(N), [\eta] \in \mathrm{CF}^{k+1}(N)$. Since the index of this bundle is -1, for a fixed metric there will be no instantons joining these two critical points. However if we consider an entire family of metrics, we expect to encounter a finite set. Of course the compactness and regularity results need to be extended in order to account for this case. Then the chain homotopy identity is again shown by equating its coefficients with the number of boundary points of a suitable one-dimensional compact manifold.

4 Finally, let us prove the functorial properties. If G is the product metric $G^2 = dt^2 + g^2$, then translation invariance holds for any instantons over this Riemannian tube. Since the bundle has index zero, this means that the instantons are constant paths in $\mathscr{B}^*(N)$. In other words $\langle \rho, \eta\rangle = [\rho = \eta]$, and so $\mathrm{CF}(g, g) = \mathrm{id}_{\mathrm{CF}^\bullet}$.

Suppose next that we have three metric g_0, g_1 and g_2. To conclude the independence of the metric, we need to show that:

$$\mathrm{CF}(g_0, g_2) = \mathrm{CF}(g_1, g_2) \circ \mathrm{CF}(g_0, g_1),$$

or in components that $\sum_{\mu(\eta)=k} |\mathscr{M}_{P(\rho,\eta)}||\mathscr{M}_{P(\eta,\tau)}| = |\mathscr{M}_{P(\rho,\tau)}|$ for $[\rho], [\tau] \in \mathrm{CF}^k(N)$. But from the gluing theorem 4.14, since we can take $S_i = \mathscr{M}_{P(\cdot,\cdot)}$, we have a map:

$$\tau_T : \mathscr{M}_{P(\rho,\eta)} \times \mathscr{M}_{P(\eta,\tau)} \to \mathscr{M}_{P(\rho,\tau)},$$

which for large T is a diffeomorphism onto its image (i.e. injective), and also surjective, by the compactness up to broken trajectories. Thus it is a bijection. From this the formula for the components follows.

Independence of the perturbation

Independence of the admissible perturbation $\underline{\sigma}^{\varepsilon}$ can be shown in a similar way, namely by defining a chain map $\mathrm{CF}(\underline{\sigma}_0^{\varepsilon}, \underline{\sigma}_1^{\varepsilon})$ that satisfies all the same formal properties as the corresponding chain map for the metric. Notice that now the auxiliary data is a homotopy $\underline{\sigma}_t^{\varepsilon}$ between the two perturbations, by which instanton equation along the tube is altered. More precisely, let $x \in \mathrm{Crit}(\mathrm{CS} + \underline{\sigma}_0^{\varepsilon})$ and $y \in \mathrm{Crit}(\mathrm{CS} + \underline{\sigma}_1^{\varepsilon})$. Then in order to count the contribution of y in $\mathrm{CF}(\underline{\sigma}_0^{\varepsilon}, \underline{\sigma}_1^{\varepsilon})[x]$ count the solutions to the equation:

$$\frac{d\omega(t)}{dt} = - \star_3 \left(\mathrm{F}_{\omega(t)}\right) - \nabla \underline{\sigma}_t^{\varepsilon}.$$

All the steps can be performed in essentially the same way as before, with slight alterations. The composition property follows by composing homotopies with a common end-point. Considering the identity property, it is important to note that ϑ is and stays an isolated critical point during any Morse homotopy of sufficiently small admissible perturbations. Thus the only flow-line starting or ending at the trivial connection is given by the four-dimensional trivial connection. We illustrate the problems arising from the trivial connection on the following example:

Example 4.24 *Consider the real plane $\mathbb{R}^2$ with the usual circle action. Define the invariant function $\Phi(z) = |z|^4$, which has a single degenerate critical point at 0. For $\varepsilon \in \mathbb{R}$ consider $\Phi_\varepsilon(z) = |z|^4 + \varepsilon|z|^2$, which for $\varepsilon \neq 0$ has a non-degenerate critical point at $z = 0$. If $\varepsilon > 0$, this is the only critical point, but if $\varepsilon < 0$, it admits a critical ring of radius $\sqrt{\frac{-\varepsilon}{2}}$. After quotienting, the induced functions on $\mathbb{R}_{>0}$ have no critical points and a single non-degenerate critical point respectively. Thus perturbing the functional Φ can lead to a situation where the perturbation has either no or a single critical point on $\mathbb{R}_{>0}$. Thus the Morse homologies of $\mathbb{R}_{>0}$ defined by these two perturbations are clearly different. The problem that arises when one tries to define a continuation map from Φ_ε to $\Phi_{-\varepsilon}$ is that the critical point merges with the critical point at the origin, which from the point of view of the quotient $\mathbb{R}_{>0}$ is not visible.*

If ϑ was not already non-degenerate for the initial Chern-Simons functional, we could potentially choose two different perturbations, one splitting off a critical point from ϑ (corresponding to the case $\varepsilon < 0$) and one which does not (corresponding to $\varepsilon > 0$). By choosing such perturbations that only differ in a neighborhood of the trivial connection we can achieve that one of the chain complexes has a single additional critical point, thus altering its Euler characteristic. Hence the Floer homology cannot be well-defined in this situation. Now the condition that ϑ is non-degenerate for CS is equivalent to requiring $\mathrm{H}^1(N) = 0$ by example 3.28, thus it was of utmost importance to require N to be a homology sphere. However since ϑ is in fact non-degenerate, we can choose our perturbations small enough such that it stays non-degenerate, and no other critical points comes close enough be sucked in. This is the reason that the Floer homology groups are well-defined in this case.

4.5.2 Examples

It is unfortunately very difficult to explicitly compute the Floer homology groups of three-manifold, since both the index and the boundary operator are highly non-trivial to determine. In fact, most known computations rely on the fact that the boundary vanishes identically for dimensional reasons. We will nonetheless give two examples.

1 The **standard three-sphere** S^3. Since $\pi_1(S^3) = 0$, the only flat connection that S^3 admits is the trivial connection, which is already non-degenerate, since $\mathrm{H}^1(S^3, \vartheta) = 0$. Thus the Floer groups have no generators, and the boundary operator is irrelevant. We have:

$$\boxed{\mathrm{HF}^\bullet(S^3) = 0.}$$

2 The **Poincaré homology sphere** P. It was the first counter-example of Poincaré's initial conjecture, which stated that any homology three-sphere must be in fact homeomorphic to the standard sphere. After he discovered P, Poincaré revised his conjecture to the modern formulation, that every connected, simply connected three-manifold must in fact be homeomorphic to S^3. It can be described as the quotient of a dodecahedron, where opposite faces are glued after rotating them by $\frac{2\pi}{5}$. Its fundamental group has order 120, so we expect it to admit non-trivial flat connections. In fact it admits two, of index 1 and 5 respectively. Thus:

$$\boxed{\mathrm{HF}^\bullet(P) = (0, \mathbb{Z}_2, 0, 0, 0, \mathbb{Z}_2, 0, 0).}$$

There are computational tools for special classes of homology spheres (see for instance 6.4 in [11]), which give rise to some further example computations. Note that the computations above are consistent with the Casson invariants of S^3 and P, as we have:

$$\lambda(S^3) = 0 = \frac{1}{2}\chi(\mathrm{HF}^\bullet(S^3)); \quad \lambda(P) = -1 = \frac{1}{2}\chi(\mathrm{HF}^\bullet(P)).$$

4.5.3 TQFT structure

We conclude our discussion of the instanton Floer homology groups by describing their $(3+1)$-dimensional **topological quantum field theory (TQFT)** structure. Recall that an n-dimensional TQFT is a covariant functor:

$$\mathscr{A} : \mathbf{nCob} \to \mathrm{Vect}_k$$

from the category of n-dimensional cobordisms (with objects $(n-1)$-manifolds and morphisms compact oriented cobordisms between them) into the category of vector spaces, such that the product identity $\mathscr{A}(N_1 \sqcup N_2) = \mathscr{A}(N_1) \otimes \mathscr{A}(N_2)$ holds. In other words, a full $(3+1)$-TQFT assigns to each three-manifold a vector space, and to each oriented cobordism (i.e. with an *in-boundary* and *out-boundary*) a linear map between the incoming and the outgoing vector space. In our case, we will consider a reduced TQFT structure on the sub-category of **4Cob** generated by disjoint unions of homology three spheres, and cobordisms which satisfy certain topolgical properties. To be precise, we define for $N = \bigsqcup_{i=1}^{n} N_i$:

$$\mathrm{HF}^{\bullet}(N) = \bigotimes_{i=1}^{n} \mathrm{HF}^{\bullet}(N_i),$$

considered as the $\mathbb{Z}_8$-graded tensor product complex. We define $\mathrm{HF}^{\bullet}(\emptyset) = \mathbb{Z}_2$. We will also consider the tensor complex on the chain level $\mathrm{CF}^{\bullet}(N) = \bigotimes_{i=1}^{n} \mathrm{CF}^{\bullet}(N_i)$. The map induced from a cobordism $M : N \Rightarrow N'$ (i.e. a compact oriented manifold M with boundary $\partial M = \bar{N} \sqcup N'$) will defined by counting instantons. Let the boundary of M be of the form $N = \bigsqcup_{i=1}^{n} N_i$ and $N' = \bigsqcup_{j=1}^{k} N'_i$, where N_i and N'_j are homology three-spheres, and adjoin semi-infinite tubes to the boundary components to make it a proper tubular manifold. We define a map between the tensor chain complexes:

$$\begin{aligned} \mathrm{HF}^{\bullet}(M) : \mathrm{CF}^{k}(N) &\to \mathrm{CF}^{k}(N') \\ [\underline{\rho}] &\mapsto \sum_{[\underline{\tau}] \in \mathrm{CF}^{k}(N')} |\mathscr{M}_{P(\underline{\rho},\underline{\tau})}| [\underline{\tau}]. \end{aligned}$$

Here $[\underline{\rho}] = [\rho_1] \otimes \cdots \otimes [\rho_n]$ and $[\underline{\tau}] = [\tau_1] \otimes \cdots \otimes [\tau_m]$ are generators of the vector space in degree k, so that $\mathscr{M}_{P(\underline{\rho},\underline{\tau})}$ is the adapted of index 0 over M, where the limits are prescribed by $[\underline{\rho}]$ and $[\underline{\tau}]$. Here it is important to note that the Fredholm index of a bundle over M can in general be expressed through the degrees of $[\underline{\rho}]$ and $[\underline{\tau}]$ in their respective tensor complexes via:

$$\operatorname{ind} \mathscr{M}_{P(\underline{\rho},\underline{\tau})} = \deg[\underline{\rho}] - \deg[\underline{\tau}] = \sum_{i=1}^{n} \mu([\rho_i]) - \sum_{j=1}^{k} \mu([\tau_j]).$$

Indeed, note that a cobordism M is composed of a tube for each boundary component as well as a center-piece which connects them. Because of the invariance of the index, we can arrange that each tube passes trough a trivial three-dimensional connection before joining the center-piece. Thus:

$$\operatorname{ind} \mathscr{M}_{P(\underline{\rho},\underline{\tau})} = \operatorname{ind} \mathscr{M}_{P(\underline{\vartheta},\underline{\vartheta})} + \sum_{i=1}^{n} \mu([\rho_i], [\vartheta]) + \sum_{j=1}^{k} \delta([\vartheta], [\tau_j]).$$

Now it is clear that $\operatorname{ind} \mathscr{M}_{P(\underline{\vartheta},\underline{\vartheta})} = 0 \in \mathbb{Z}_8$ by considering the four-dimensional trivial connection as an example of an adapted connection. The identity we claimed now follows from $\delta([\rho], [\eta]) = \mu([\rho]) - \mu([\eta])$ and $\mu([\vartheta]) = 0$. What is left to be checked is that the moduli space $\mathscr{M}_{P(\underline{\rho},\underline{\tau})}$, which we know is generically a discrete set of points, is compact, that the map above is indeed a chain map, and that on the level of homology it is independent of the auxiliary metric chosen on M. These verifications are very similar to what we had to check in section 4.5.1. When analysing the compactness property in particular, it is important that a chain of instantons on M cannot factor trough a reducible connection on the center-piece (since compactness requires the limit chain to be composed of irreducible terms). So the class of cobordisms we can consider must be such that this cannot happen.

Example 4.25 *Let N_0, N_1 be two homology three-spheres, and let $M : N_0 \Rightarrow N_1$ be a cobordism with* $\mathrm{H}^1(M) = 0$*. We claim that no sequence in $\mathscr{M}_{P(\underline{\rho},\underline{\tau})}$ can converge to a chain with reducible terms. Note that by our assumption on homology, the only reducible instanton on M must be the trivial connection. By formula 4.17 we have for such a limit chain $(\underline{P}, \underline{Z})$ that:*

$$0 = \operatorname{ind}\Theta + \dim \mathrm{H}^1(N_0, \vartheta) + \dim \mathrm{H}^1(N_1, \vartheta) + \operatorname{ind} P_- + \operatorname{ind} P_+ + 8k.$$

By general position, the indices of the limiting tubes on both ends must be non-negative (since it supports an instanton), thus:

$$0 \geq \operatorname{ind}\Theta + \dim \mathrm{H}^1(N_0, \vartheta) + \dim \mathrm{H}^1(N_1, \vartheta).$$

However computation shows that $\dim \mathrm{H}^1(N_i, \vartheta) = 3$ *and* $\operatorname{ind}\Theta = -3$*, leading to the absurd conclusion* $0 \geq 3$*. Thus in this case no reducible instantons can appear on the center-piece of the cobordism. A similar reasoning shows that this cannot happen on the tubes either, so that the moduli space is indeed compact.*

Next, we need to verify the functorial property of the TQFT given by instanton Floer homology. This follows again closely our description of the functoriality with regards to the metric. In particular if $M : N_0 \Rightarrow N_1$ and $M' : N_1 \Rightarrow N_2$ are two cobordisms, such that $\partial_{out} M = \partial_{in} M'$, we can compose them to yield a cobordism $M \circ M' : N_0 \Rightarrow N_2$. In particular, by sufficiently stretching the necks, we can achieve that all instantons over $M \circ M'$ are the result of gluing, and the gluing map then provides a bijection of finite sets:

$$\mathscr{M}_{P^\sharp(\underline{\rho_0},\underline{\rho_2})} \simeq \mathscr{M}_{P(\underline{\rho_0},\underline{\rho_1})} \times \mathscr{M}_{P(\underline{\rho_1},\underline{\rho_2})}.$$

This proves the composition property. Finally the product property (where the disjoint union of manifolds goes over into the tensor product of vector spaces) is evident from the definition of our TQFT. The utility of this description is that it provides invariants for closed four-manifolds M, seen as cobordisms between empty manifolds $M : \emptyset \Rightarrow \emptyset$, and these invarints can in principle be computed by cutting the four-manifold along a three-dimensional slice. For instance, if a M admits a embedded sphere $S^3 \subset M$, which separates M into two components $M = M_1 \cup M_2$, then the invariant so defined must vanish. Indeed, say $\partial M_1 = \bar{\emptyset} \sqcup S^3$ and $\partial M_2 = \bar{S}^3 \sqcup \emptyset$, then by functoriality:

$$\mathrm{HF}^\bullet(M) = \mathrm{HF}^\bullet(M_1) \circ \mathrm{HF}^\bullet(M_2) : \mathbb{Z}_2 \to \mathrm{HF}^\bullet(S^3) \to \mathbb{Z}_2.$$

Thus, since $\mathrm{HF}^\bullet(S^3) = 0$, the invariant vanishes in this case. Since a manifold can potentially be cut along many embedded three-manifolds, we see that the TQFT-structure on the Floer homology groups is a restrictive condition. As an example, 2-dimensional TQFTs are equivalent to a very restrictive class of algebras, so callled **Frobenius algebras** (see e.g. [13]).

Appendix A

Analysis

In this appendix we introduce the the analysis of partial differential operators on manifolds which we require in our examination of the instanton equation. All manifolds, vector bundles and principal bundles are assumed smooth. All manifolds will be oriented.

A.1 Sobolev spaces of sections

Let $E \to M$ be a rank k vector bundle over a manifold (possibly with boundary), and let $\Gamma(E)$ denote its smooth sections. We would like to define a notion of smooth structure on this space (or an extension of it) which is well behaved under the usual analytic operations such as taking limits. We present two possible answers: **Fréchet** space structure and **Sobolev** space structure. Consider first the example of the closed unit ball $\mathbb{D}^n \subset \mathbb{R}^n$ and a trivial line bundle L over it. Then we have the isomorphism $\Gamma(L) \simeq C^\infty(\mathbb{D}^n)$, the latter of which is a Fréchet space with respect to the family of semi-norms $\{\sum_{i=0}^k ||\cdot||_{C^i}\}_{k\in\mathbb{N}}$. However Fréchet spaces are analytically hard to work with, since they for instance do not satisfy the implicit function theorem. Hence we would prefer a Banach or, if possible, a Hilbert space structure. In our model a possible answer is given by the Sobolev spaces $W^{k,p}(\mathbb{D}^n)$ of order $k \in \mathbb{N}$ and regularity $p \in [1, \infty]$, where the Hilbert case is given if $p = 2$. Let us now extend these definitions from the toy model onto possibly non-trivial and higher rank vector bundles.

First the Fréchet space structure. For this we need a countable family of semi-norms. We choose the following:

Definition A.1 *Let $K = \bar{U}$ be an compact subset which is the closure of an open subset of M and $\varphi : E|_K \to \mathbb{R}^k$ a vector bundle chart over K. We define the semi-norm:*

$$||s||_{l,K} = ||\varphi \circ s|_K||_{C^l(K)}$$

Let now $\{K_i, \varphi_i\}_{i\in\mathbb{N}}$ be a trivialization of E. We then have a countable family of semi-norms given by $\|\cdot\|_{l,i} = \|\cdot\|_{l,K_i}$. Since $\mathbb{N}\times\mathbb{N}$ is countable we can re-index them as $\|\cdot\|_j$ for $j \in \mathbb{N}$. Then define the increasing family of semi-norms

$$\|\cdot\|_{F,k} = \sum_{j=0}^{k} \|\cdot\|_j$$

We are now in a position to define a Fréchet space structure analogous to the one on $C^\infty(\mathbb{D}^n)$ by the following proposition.

Proposition A.2 (Fréchet space of sections) *The space $\Gamma(E)$ together with the countable family of semi-norms $\|\cdot\|_{F,k}$ is a Fréchet space, and the Fréchet space structure does not depend of the ordering of $\mathbb{N}\times\mathbb{N}$. Furthermore, if M is compact it is also independent of the family of trivializations chosen.*

Proof Let $s \in \Gamma(E)$ be a smooth section, and suppose $\|s\|_{F,k} = 0$ for all k. But then in particular $\|s|_{K_i}\|_{C^0(K_i)} = 0$ for all i, and hence $s = 0 \in \Gamma(E)$, so the family of semi-norms is non-degenerate. It remains to show completeness with respect to the family of semi-norms. Suppose therefore that $\{s_n\}_{n\in\mathbb{N}} \subset \Gamma(E)$ is a sequence of sections which is Cauchy with respect to all the $\|\cdot\|_{F,k}$. By unravelling the definitions we get $\|s_n|_{K_i}\|_{C^k(K_i)}$ is Cauchy. But then since $C^\infty(K_i)$ is Fréchet, there is $t_i \in \Gamma_{K_i}(E)$ such that $s_n|_{K_i} \to t_i$ smoothly. The t_i must agree on their overlap, an hence fit smoothly together to form a section $t \in \Gamma(E)$. It is then clear that $s_n \to t$ w.r.t. the family of semi-norms $\|\cdot\|_{F,k}$, and so $\Gamma(E)$ has a Fréchet space structure, induced by the family of semi-norms.

Now, this Fréchet structure is independent of the ordering, for let $\eta : \mathbb{N} \to \mathbb{N}$ be a bijection, and let

$$\|\cdot\|'_{F,k} = \sum_{j=0}^{k} \|\cdot\|_{\eta(j)}.$$

But then $\|\cdot\|'_{F,k} \leq \|\cdot\|_{F,\max_{i=0,\dots,k}\eta(i)}$ and similarly $\|\cdot\|_{F,k} \leq \|\cdot\|'_{F,\max_{i=0,\dots,k}\eta^{-1}(i)}$, so the two structures are equivalent.

Finally, let M be compact. We will now show that in this the Fréchet structure does not depend on the trivializations chosen. First, consider two different trivializations $\varphi_1, \varphi_2 : E|_K \to \mathbb{R}^m$ over a compact subset $K \subset M$, with transition function $g : K \to \mathrm{GL}(m) \subset \mathbb{R}^{m\times m}$. Let $s \in \Gamma_K(E)$ be a local section. Then for $I \in \mathbb{N}^n$ a multi-index:

$$\begin{aligned}
\|\varphi_1 \circ s\|_{C^k(K)} =& \|g(\varphi_2 \circ s)\|_{C^k(K)} \\
\leq& \sum_{i=0}^{k} \|g\|_{C^{k-i}(K)} \|\varphi_2 \circ s\|_{C^i(K)} \\
\leq& C \sum_{i=0}^{k} \|\varphi_2 \circ s\|_{C^i(K)}
\end{aligned}$$

Now by this bound we can again find that the two Fréchet structures with one trivialization replaced by another are equivalent. In fact one can replace any number of trivializations, since each semi-norm $\|\cdot\|_{F,k}$ only has finitely many terms. Finally, it is easy to see that the Fréchet structure induced by a refinement of a cover is the same as the one induced by the original cover, since the original cover can be covered by finitely many compact sets of the new cover, as the compact sets constituting the cover contain a a non-empty open set. □

Note that the Fréchet space structure not necessarily unique on non-compact manifolds. However, the above uniqueness result can be extended to the case of tubular manifolds by restricting to product sets over the ends. Let us finish our discussion of Fréchet spaces by providing a useful lemma:

Lemma A.3 (Smooth exponential law) *Let N be a compact manifold. Then:*

$$C_c^\infty(\mathbb{R} \times N) \simeq C_c^\infty(\mathbb{R}, C^\infty(N)),$$

as Fréchet spaces.

Proof To $\Phi \in C^\infty(\mathbb{R}, C^\infty(N))$ we associate the map $\varphi \in C^\infty(\mathbb{R} \times N)$ given by $\varphi(t,x) = \Phi(t)(x)$. Similarly to $\varphi \in C^\infty(\mathbb{R} \times N)$ we associate $\Phi \in C^\infty(\mathbb{R}, C^\infty(N))$ by the same formula. That this bijective assignment of sets restricts to the compactly supported forms is clear, since compactly supported on $\mathbb{R} \times N$ means non-zero on a bounded subset of the $\mathbb{R}$-factor.

Recall that a map $\Phi : \mathbb{R} \to C^\infty(N)$ is continuous at a time $t \in \mathbb{R}$ exactly when $t_i \to t$ implies $\|\Phi(t) - \Phi(t_i)\|_{C^k(N)} \to 0$ for all k simultaneously. Thus by compactness of N, continuity of Φ and φ are equivalent. For the higher derivatives, the following identity is key:

$$\frac{d}{dt}\Phi(t)(x) = (\partial_t\varphi)(t,x),$$

for it implies that $\dot{\Phi}$ is continuous iff $\partial_t\varphi$ is. Since φ is always continuously differentiable in the space directions by assumption on Φ, we have established the correspondence. □

We are however not going to dive into Fréchet space theory any further, as the analytically better behaved Sobolev space structure is more suited to our purposes. In analogy to the case of the trivial line bundle, in order to define it we need a scalar product of sections as well as a derivative for sections. For the first we use an auxiliary metric m on E as well as a Riemannian metric g on M, and for the second we use a fixed background connection. We then show that these additional data do not impact the resulting Banach space structure, at least in the compact and tubular cases.

Definition A.4 (L^p space of sections) *Let $(E,m) \to (M,g)$ be a metric vector bundle over a Riemannian manifold. The metrics induce a scalar product on $\Gamma(E)$, which for sections $s,t \in \Gamma(E)$ is given by:*

$$\langle\langle s,t \rangle\rangle = \int_{M,g} \langle s(x), t(x) \rangle_{E_x}.$$

*The L^p-**norm of a section** $s \in \Gamma(E)$ is then defined by:*

$$\|s\|_p = \langle\!\langle s, s \rangle\!\rangle^{p/2}.$$

We define the space of L^p-sections of E as the closure of the compactly supported sections w.r.t. the L^p-norm:

$$L^p(E) = \|\cdot\|_p - \operatorname{clos} \Gamma_c(E).$$

Proposition A.5 *Let M be a compact base manifold or a tube $N \times \mathbb{R}$ with a product metric $G^2 = g^2 + dt^2$. Then the definition of $L^p(E)$ is independent of the choices of m and g.*

Proof We will prove the compact case first. First, the choice of Riemannian metric does not matter, for if g' is a different metric, and since on finite-dimensional vector spaces all inner products are equivalent, there is a continuous function $C : M \to \mathbb{R}_{>0}$ such that $\frac{1}{C(x)} g_x \leq g'_x \leq C(x) g_x$ as bilinear forms. Now by compactness, C has a maximum $C_{\max}$, for which $\frac{1}{C_{\max}} g_x \leq g'_x \leq C_{\max} g_x$ holds for all $x \in M$. We thus get for $f \in C^\infty(M)$:

$$(C_{\max})^{-n/2} \int_{M,g} f \leq \int_{M,g'} f \leq (C_{\max})^{n/2} \int_{M,g'} f$$

and thus that the two norms induced by these different metrics are equivalent. Essentially the same argument gives independence of m.

Now in the tubular case, M is no longer compact, however still have $\frac{1}{C(x)} g_{x,t} \leq g'_{x,t} \leq C(x) g_{x,t}$, owing to the fact that on the $\mathbb{R}$-factor both metrics agree. So the same argument by compactness goes through. Noting the same for m yields independence of all additional data. $\square$

We have now shown that $L^p(E)$ is independent of the metrics on E and TM in the compact case and in the case of tubes with given metric on the $\mathbb{R}$-factor. In the general paracompact case however, this is not necessarily the case anymore, as for instance one can imagine two Riemannian structures on M, one with finite volume and the other with infinite volume. If the bundle is trivial, then a non-zero constant section will be in the first L^p-space, but not in the second. Since for a general vector bundle, some sections are non-zero on an open-dense subset by transversality theory, similar examples can be constructed in the non-trivial case.

Next, we will introduce the higher Sobolev norms on vector bundles in analogy to the trivial line bundle on $\mathbb{D}^n$. Recall that for $f \in C^\infty(\mathbb{D}^n)$ these are given by:

$$\|f\|_{k,p} = \sum_{|J| \leq k} \|\partial x^J f\|_p.$$

In order to generalize we introduce as additional data a background connection $\nabla \in \mathscr{C}(E)$ as well as its **higher covariant derivatives** acting on $s \in \Gamma(E)$ as:

$$\begin{aligned}
\nabla^2(s) &= (\nabla_{LC} \otimes \nabla)(\nabla s) \\
\nabla^3(s) &= (\nabla_{LC}^2 \otimes \nabla)(\nabla^2 s) \\
&\dots \\
\nabla^{k+1}(s) &= (\nabla_{LC}^k \otimes \nabla)(\nabla^k s).
\end{aligned}$$

Here ∇^{LC} is the Levi-Civita connection on (M, g), meaning that the higher covariant derivatives are maps $\nabla^k : \Gamma(E) \to \Gamma(\bigotimes^k T^*M \otimes E)$. We can then define the higher Sobolev norms as:

$$\|s\|_{k,p} = \sum_{j \leq k} \|\nabla^j s\|_p,$$

Where the L^p norms are the ones on $\bigotimes^k T^*M \otimes E$. The term $\|\nabla^j s\|_p$ should be understood as the generalisation of $\sum_{|J|=j} \|\partial x^J f\|_p$, and indeed if we choose the trivial connection as the background connection, both are equal. This leads us to the definition of general Sobolev spaces on manifolds.

Definition A.6 (Sobolev space of sections) *Let $k \in \mathbb{N}, 1 \leq p < \infty$. Then:*

$$W^{k,p}(E) = \|\cdot\|_{k,p} - \operatorname{clos} \Gamma_c(E)$$

As before it can be shown that in the tubular case this definition is not dependent on the metrics on both M and E, and in addition to that it is also independent of the background connection chosen. The latter fact follows from the affine structure of the space of connections. Let $\nabla_1, \nabla_2 \in \mathscr{C}(E)$ be two connections. Then $\nabla_2 - \nabla_1 = A \in \Gamma(\mathrm{End}(E))$ and so $\|\nabla_2 s\|_p \leq \|\nabla_1 s\|_p + C\|s\|_p$, where $C > 0$ is a constant depending on A. Here is it important that M be tubular, since otherwise a global constant C does not necessarily exist. Since similar estimates hold for higher covariant derivatives as well as when the two connections are interchanged, the Banach space structures w.r.t. these connections agree.

Let us now give a few examples of Sobolev spaces that we will need:

Example A.7

- *Consider the space of l-forms on M, i.e. sections of $\Lambda^l T^*M$. We will denote Sobolev spaces of l-forms by:*

  $$W^{k,p}(\Omega^l(M)) = W^{k,p}(\Lambda^l T^*M).$$

  Similarly we will write the Sobolev space of l-forms with coefficients in a bundle E as:

  $$W^{k,p}(\Omega^l(M,E)) = W^{k,p}(\Lambda^l T^*M \otimes E).$$

- *We can give the space of connections on a principal bundle P a Sobolev structure by defining:*

$$\mathscr{C}^{k,p}(P) := \varpi_0 + W^{k,p}(\Omega^1(M, \operatorname{ad} P)).$$

Here ϖ_0 is a smooth reference connection, whose choice is irrelevant for the definition if the base manifold is compact. In the non-compact case this is no longer the case, and the function spaces introduced will only include sections which satisfy given boundary conditions (see the analytical treatment in chapter 4).

Note that covariant derivatives on Sobolev spaces behave in very much the same way as the usual time derivative, seen as an operator $\frac{d}{dt} : W^{k+1,p}(\mathbb{R}) \to W^{k,p}(\mathbb{R})$.

Proposition A.8 *Let $E \to M$ be a vector bundle and let $\nabla \in \mathscr{C}(E)$ be a connection. Then the exterior covariant derivative $d^\nabla : \Omega^\bullet(M, E) \to \Omega^{\bullet+1}(M, E)$ extends to a map:*

$$d^\nabla : W^{1,p}(\Omega^\bullet(M, E)) \to L^p(\Omega^{\bullet+1}(M, E)).$$

Proof Use ∇ as the background reference connection on E, then the result follows immediately. Since the Sobolev structure is independent of the background connection, this choice is justified. □

On manifolds, as on Euclidean domains there are different avenues to constructing L^p and $W^{k,p}$-spaces. We have here used the density of test functions in Sobolev spaces for our definition. But there are different options, which we will now shortly sketch. On compact manifolds for instance, a Sobolev section can be given in terms of local Sobolev representatives defined on charts. This is because the only obstruction to integrability on compact manifolds is the presence of singularities. When we moved on to higher Sobolev spaces, we could have defined them in terms of distributional derivatives with respect to some covariant derivative, i.e. as the space of L^p-sections that have a certain number of covariant derivatives in L^p as well. Both these viewpoints are equivalent by the results from Sobolev theory on Euclidean domains. Finally one can go one step further on manifolds and consider weak derivatives with respect to first order **partial differential operators** (to be defined precisely in A.5) that are not covariant derivatives. Consider for instance the deformation operator $D_\varpi = (-\delta_\varpi) \oplus d_\varpi : \Omega^1(\operatorname{ad} P) \to \Omega^{0,2^+}(\operatorname{ad} P)$. There is a so called **Weitzenböck formula** relating it to covariant derivatives:

$$D_\varpi^* D_\varpi = \nabla^* \nabla + R.$$

In this expression R is a 0-th order operator, i.e. an endomorphism. It is a fact that on tubular manifolds the operator norm of R_x is uniformly bounded. From this we see that the $W^{1,2}$-norm on $\Omega^1(\operatorname{ad} P)$ can equally well be defined with respect to D_ϖ and provides an equivalent Banach space structure. This has the immediate application:

Corollary A.9 *The operator*

$$D_{\varpi} : W^{1,2}(\Omega^1(\operatorname{ad} P)) \to L^2(\Omega^{0,2^+}(\operatorname{ad} P))$$

is a bounded linear operator.

Proposition A.10 *Let $\varpi \in \mathscr{C}^{k,p}(P)$ be a Sobolev connection with $k > \frac{n}{p}$. Then* $\mathrm{F}_{\varpi} \in W^{k-1,p}(\Omega^2(M, \operatorname{ad} P))$.

Proof Work in local coordinates, where $\mathrm{F}_{\mathbf{A}} = d\mathbf{A} + \mathbf{A} \wedge \mathbf{A}$. Now the result follows from A.8 and the Sobolev multiplication theorem (see for instance theorem 4.4.1 in [5]). □

To conclude this section, let us investigate how the Sobolev and Fréchet structures interact

Proposition A.11 (Relation between Fréchet and Sobolev structure) *Let $E \to M$ be a vector bundle over a compact manifold. The inclusion of Fréchet spaces $\Gamma(E) \hookrightarrow W^{k,p}(E)$ is continuous.*

Proof For this one needs to show that $||s||^{k,p} \leq C||s||_i$, where $i \in \mathbb{N}$ is a sufficiently big semi-norm for the Fréchet space structure and $C > 0$. Notice that for $s \in \Gamma(E)$ we have due to compactness:

$$||s||^{k,p} = \sum_{j \leq k} \left(\int_{M,g} |\nabla^j s|^{\frac{1}{p}} \right)^p \leq C \sup_{j \leq k, x \in N} |\nabla^j s(x)|$$

And the last expression can be bounded independently of any additional data by Fréchet semi-norms of high enough order. □

A.2 Sobolev spaces of gauge transforms

The space of gauge transformations $\mathscr{G}(P) \subset \operatorname{Diff}(P)$ is a topological group when endowed with the compact-open topology. In this section we will give it smooth structures in the case where the gauge group G is a matrix Lie group, i.e. $G \subset \mathrm{GL}(n, \mathbb{K})$ with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. This will be most relevant for our purposes, since we will be working exclusively with $\mathrm{SU}(2)-$bundles. We essentially follow Morgan (see [5][4.4]). Recall that $\mathscr{G}(P) \simeq \Gamma(\operatorname{Ad} P)$. There is an embedding of that latter fibre bundle into a vector bundle:

$$\operatorname{Ad} P = P \times_c G \hookrightarrow P \times_G \mathbb{K}^{n \times n}.$$

Here G acts on matrices by conjugation, making the inclusion well-defined. Let $k \geq 0, 1 \leq p < \infty$ such that $k > \frac{n}{p}$. We then define the space of **Sobolev gauge transformations** $\mathscr{G}^{k,p}(P)$ as the subset of $W^{k,p}$-sections of $P \times_G \mathbb{K}^{n \times n}$ which are also sections of Ad. Our assumption on k and n guarantees that these sections will all be continuous.

Proposition A.12 (Smooth action, lemma 4.4.3 in [5]) *Let n, k be such that $k+1 > \frac{n}{p}$. Then $\mathscr{G}^{k+1,p}(P)$ is a Banach Lie group with Lie algebra $\Gamma^{k+1,p}(\operatorname{ad} P)$. The group acts smoothly on the space of $W^{k,p}$-connections $\mathscr{C}^{k,p}(M)$.*

In our treatment in four dimensions we will follow the convention of Morgan (see [5]) work with $\mathscr{G}(P) = \mathscr{G}^{3,2}(P)$, $\mathscr{C}(P) = \mathscr{C}^{2,2}(P)$ and $W^{1,2}(\Omega^2(M, \operatorname{ad} P))$, as in this scenario both the gauge transformations and connections are continuous, the action is smooth, and we work in the category of Hilbert manifolds.

For our three-manifolds N we will follow the convention of the original paper by Floer (see [7]) and work with $\mathscr{G}(Q) = \mathscr{G}^{2,4}(Q)$, $\mathscr{C}(Q) = \mathscr{C}^{1,4}(Q)$ and $L^4(\Omega^2(N, \operatorname{ad} Q))$. Here again the gauge transformations and connections are continous with smooth action.

A.3 Fredholm operators

The space of instantons can be seen as the pre-image of a point under a smooth map between Banach spaces, namely the self-dual part of the curvature F^+ on well-chosen Sobolev completions of $\mathscr{C}(P)$ and $\Omega^2_+(\operatorname{ad} P)$. In the construction of the Floer homology (and of gauge-theoretic invariants in general), it is important that the instantons form finite-dimensional smooth manifolds. This means that for $\varpi \in \mathscr{C}(P)$ an instanton, the differential $d\,\mathrm{F}^+(\varpi)$ needs to be a surjective linear map with finite-dimensional kernel. In this section we will investigate linear maps which are a slight generalisation of this concept, the **Fredholm operators**. More precisely, a bounded linear operator $A : X \to Y$ between Banach spaces is **Fredholm** if:

- $\ker A$ is finite-dimensional.
- $\operatorname{coker} A := Y/\operatorname{im} A$ is finite-dimensional.

In other words, while Fredholm operators are not necessarily surjective, they only miss a finite dimensional portion of their target space. We denote the set of Fredholm operators between Banach spaces X and Y by $\operatorname{Fred}(X, Y) \subset \mathscr{L}(X, Y)$. Since both the kernel and the cokernel of a Fredholm operator are finite-dimensional, there is a well-defined **Fredholm index**:

$$\operatorname{ind} : \operatorname{Fred}(X, Y) \to \mathbb{Z}; \quad A \mapsto \dim \ker A - \dim \operatorname{coker} A$$

This index can be seen as a measure of non-bijectivity. If it is positive, then the operator cannot be injective, and if it is negative, the operator cannot be surjective. And in the case where an operator is surjective, its index is exactly the dimension of its kernel. Thus if 0 is a regular value of F^+, the dimension of the space of instantons will be given by $\operatorname{ind} d\,\mathrm{F}^+$. We now present the most important properties of Fredholm operators and their index.

Proposition A.13 *The set of Fredholm operators $\operatorname{Fred}(X, Y) \subset \mathscr{L}(X, Y)$ is open.*

The index is easier to work with than the dimension of the kernel, because the latter can jump discontinuously when perturbing an operator. Just consider multiplication by a scalar $\lambda \in \mathbb{R}$ as a map $\mathbb{R} \to \mathbb{R}$, which has $\dim\ker(\cdot\lambda) = \delta_0^\lambda$, while the index is always equal to 0. To be precise we have the following results:

Proposition A.14 *The index is locally constant, meaning that if $\{A_t\}_{t\in[0,1]}$ is a homotopy of Fredholm operators through Fredholm operators which is continuous in the norm topology, then* $\operatorname{ind} A_0 = \operatorname{ind} A_1$.

Proposition A.15 *Let $A \in \operatorname{Fred}(X, Y)$ be Fredholm and $K \in \mathscr{L}(X, Y)$ be compact. Then $A + K$ is again Fredholm, with the same index.*

In fact, for a special class of Fredholm operators, the elliptic partial differential operators on compact manifolds (see section A.5), the index is solely determined by the topology of the vector bundles and manifolds involved in the form of the Atyiah-Singer Index Theorm that we will later encounter (see A.24).

We can extend these results about linear maps to general maps and reconnect with the setting of instantons. We say a C^1 map $\mathbf{F} : \mathscr{M} \to \mathscr{N}$ between C^1-Banach manifolds is a **Fredholm map**, if its derivative $d\mathbf{F}_x : T_x\mathscr{M} \to T_{\mathbf{F}(x)}\mathscr{N}$ is Fredholm for all $x \in \mathscr{M}$.

Since we can trivialize the tangent bundles around neighbourhoods of $x \in \mathscr{M}$ and $\mathbf{F}(x) \in \mathscr{N}$ respectively, and the Fredholm index is locally constant, we get a well defined index of $\mathbf{F}$ for each connected component of $\mathscr{M}$. In particular, if $\mathscr{M}$ is connected, then we can associate to $\mathbf{F}$ a well-defined index, namely the index of any one of its linearizations.

In this situation we can then apply the following Banach version of the implicit function theorem to conclude that the set of instantons is a finite-dimensional smooth manifold, given its Fredholm property.

Proposition A.16 *Let $\mathscr{M}, \mathscr{N}$ be C^1-Banach manifolds with $\mathscr{N}$ connected, and let $\mathbf{F} : \mathscr{M} \to \mathscr{N}$ be a Fredholm map. Let $q \in \mathscr{N}$ be a regular value of $\mathbf{F}$, i.e. for all $p \in \mathbf{F}^{-1}(q)$, the linear map $d\mathbf{F}(p)$ has a bounded right-inverse. Then $\mathbf{F}^{-1}(q) \subset \mathscr{M}$ is a finite-dimensional submanifold of dimension* $\operatorname{ind}\mathbf{F}$.

A.4 Symmetric and self-adjoint operators

Let W and H be two real Hilbert spaces with a compact dense inclusion $W \hookrightarrow H$. The spectrum and resolvent set of any operator $T : W \to H$ is thus a subset of the real line by definition. Operators $T : W \to H$ can then be interpreted either as bounded operators from W to H or as unbounded operators on H.

Definition A.17 *An unbounded operator $T : W \subset H \to H$ between Hilbert spaces is **symmetric**, if for all $x, y \in W$ we have $\langle Tx, y\rangle = \langle x, Ty\rangle$. It is self-adjoint if furthermore the domains of T and its adjoint T^* coincide.*

Proposition A.18 (5.12 in [14]) *Let T be an unbounded self-adjoint operator $T : W \to H$. Then the following are equivalent:*

- *There is an orthonormal basis of eigenvectors for T.*
- *The spectrum of A is discrete.*
- *The resolvent of A is compact for some and hence for all $\lambda \in \sigma(T)$.*
- *The inclusion $(W, \|\cdot\|_T) \hookrightarrow H$ is compact, where $\|\cdot\|_T$ is the graph norm.*

Notice that self-adjoint operators with discrete spectrum thus only exist on seperable Hilbert spaces. Also, we have the following criterion of self-adjointness for a closed symmetric operator:

Proposition A.19 *Let $A \in \mathscr{L}_{sym}(W, H)$ have non-empty resolvent set $\rho(A)$, i.e. there is a real $\lambda \in \mathbb{R}$ such that $A - \lambda : W \to H$ is bijective. Then A is self-adjoint.*

Proof Let $\lambda \in \rho(A)$ as above. Since A is symmetric, its adjoint A^* is an extension of A. Consider an element of the domain of this adjoint $\xi \in D(A^*) \subset H$, which means $\xi \in H$ such that $v \mapsto \langle Av, \xi\rangle$ is a continuous functional. We need to show that $\xi \in W$. Notice that $(A^* - \lambda)\xi \in H$, so by surjectivity of $A - \lambda$ there is $\eta \in W$ with:

$$(A - \lambda)\eta = (A^* - \lambda)\xi$$

This then implies for every $v \in W$:

$$\begin{aligned} \langle (A-\lambda)v, \xi\rangle =& \langle v, (A^* - \lambda)\xi\rangle \\ =& \langle v, (A-\lambda)\eta\rangle \\ =& \langle (A^* - \lambda)v, \eta\rangle \\ =& \langle (A-\lambda)v, \eta\rangle \end{aligned} \qquad \square$$

where the last line is due to the symmetry of A. Since $\operatorname{im}(A - \lambda) = H$ by assumption, we must have $\xi = \eta \in W$, thus A is self-adjoint.

Proposition A.20 *Let the inclusion $W \to H$ be compact. The set of self-adjoint operators $\mathscr{S}(W, H) \subset \mathscr{L}_{sym}(W, H)$ is open.*

Proof Let $A \in \mathscr{S}(W, H)$. Then by theorem A.18 there must be some $\lambda \in \rho(A)$, such that $A - \lambda \in \mathscr{L}(W, H)$ is an invertible operator. But being invertible is an open condition, hence a neighborhood of $A \in \mathscr{L}_{sym}(W, H)$ consists only of invertible operators, i.e. in an open neighborhood of A every operator has non-empty resolvent set. Thus by the previous lemma this neighborhood consists solely of self-adjoint operators. $\square$

A.5 Elliptic operators on manifolds

Let E^n, F^m be two $\mathbb{K}$-vector bundles (with $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) over the same base manifold M^p. A **partial differential operator** between E and F of order k is an $\mathbb{R}$-linear map $D : \Gamma(E) \to \Gamma(F)$ which in local coordinates is of the form:

$$(Ds)_\alpha(x) = \sum_{|I|\leq k} \Phi_I^\alpha(x) \frac{\partial s_\alpha}{\partial x^I}(x).$$

Here $\Phi_I^\alpha : U_\alpha \to \hom(\mathbb{K}^n, \mathbb{K}^m)$ are homomorphisms of the trivialized bundles. We do not require the Φ_I^α to be the same for every choice of local coordinates, in fact in general they will depend on the trivialization chosen. However notice that if we choose different trivializations on E and F such that $(Ds)_\beta = g^F_{\alpha\beta}(Ds)_\alpha$ and $s_\beta = g^E_{\alpha\beta} s_\alpha$ we can compute:

$$\begin{aligned} g^F_{\alpha\beta}(Ds)_\alpha(x) = (Ds)_\beta(x) &= \sum_{|I|\leq k} \Phi_I^\beta(x) \frac{\partial s_\beta}{\partial x^I}(x) \\ &= \sum_{|I|\leq k} \Phi_I^\beta(x) \frac{\partial}{\partial x^I}(g^E_{\alpha\beta} s_\alpha)(x) \\ &= \sum_{|I|=k} \Phi_I^\beta(x) g^E_{\alpha\beta}(x) \frac{\partial s_\alpha}{\partial x^I}(x) + \text{(lower order terms)} \end{aligned}$$

and hence $g^F_{\alpha\beta}\Phi_I^\alpha = \Phi_I^\beta g^E_{\alpha\beta}$ for $|I| = k$, meaning that the highest-order terms fit together to form a bundle map $\Phi_I \in \hom(E, F)$. Furthermore, let dx^i be the local trivialization of T^*M corresponding to the coordinates x^i. Consider the following local sections:

$$\begin{aligned} (\sigma_D)_\alpha : U_\alpha \times S^k(T^*U_\alpha) &\to U_\alpha \times \hom(E|_{U_\alpha}, F|_{U_\alpha}) \\ (x, dx^J) &\mapsto \sum_{|I|=k} \Phi_I^\alpha(x)[I \text{ and } J \text{ agree as unordered tuples}] \end{aligned}$$

Here $S^k(T^*U_\alpha)$ is the symmetric tensor product of k copies of T^*U_α, the notation dx^J for a multi-index $J = (J_1, \ldots, J_k)$ denotes $dx^{J_1} \otimes \cdots \otimes dx^{J_k}$ and the formula $[P]$ for a proposition P evaluates to 1 if the proposition is true, and 0 otherwise. It can be checked that $(\sigma_D)_\alpha$ transform as tensors, and so fit together to give a homomorphism: $\sigma_D : S^k(T^*M) \to \hom(E, F)$, which is called the **principal symbol** of D. Note that a homomorphism $S^k(T^*M) \to \mathrm{Hom}(E, F)$ can also be seen as a homogeneous polynomial map of degree k, by precomposing with:

$$T^*M \to S^k(T^*M); \quad \xi \to \xi \otimes \cdots \otimes \xi.$$

One can determine to the homomorphism from the homogeneous polynomial map by a procedure called **polarisation**, which extends the duality between quadratic forms an symmetric bilinear maps on vector spaces (see lecture 37 of [4]). The principal symbol has a few key properties:

Proposition A.21 *Let $P : \Gamma(E) \to \Gamma(F)$ and $Q : \Gamma(F) \to \Gamma(G)$ be partial differential operators. Then:*

- $\sigma_{Q \circ P}(\xi) = \sigma_Q(\xi)\sigma_P(\xi)$ *for $\xi \in T^*M$ in the non-polarised form.*
- *If P has an adjoint $P^* : \Gamma(F) \to \Gamma(E)$, then $\sigma_{P^*} = \sigma_P^*$*

Relevant for us are the **elliptic operators**, that is operators $D : \Gamma(E) \to \Gamma(F)$ such that for every $\xi \in T^*M$ the map $\sigma(D)(\xi \otimes \cdots \otimes \xi) : E_x \to F_x$ is a linear isomorphism. Examples include the second-order Laplacian Δ on $\mathbb{R}^n$, which has symbol $\sigma_\Delta(\xi) = \|\xi\|^2 \operatorname{id}$, as well as the first-order Dirac operator $D = \frac{d}{dt} + \lambda$ on S^1 with symbol $\sigma_D(\xi) = \|\xi\| \operatorname{id}$. For such operators a number of important properties hold.

Theorem A.22 (Fredholm property, [15] Thm. 4.8) *Every elliptic operator on a compact manifold is Fredholm.*

Note that this is in general not the case on non-compact manifolds. As an example, $\frac{d}{dt}$ is elliptic on $\mathbb{R}$, but not Fredholm. To see this we just note that its image is not closed. Indeed consider $f \in C_c^\infty(\mathbb{R}) \subset L^2(\mathbb{R})$ with $I = \int_{\mathbb{R}} f dx \neq 0$. Since every element $h = \frac{dg}{dt} \in \operatorname{im} \frac{d}{dt}$ must satisfy: $\int_{\mathbb{R}} h dx = \int \frac{d}{dt} g dx = 0$, f is not in the image of $\frac{d}{dt}$. However consider $f_n = f - \frac{I}{n}\chi_{(0,n)}$, which have integral 0, and so are in the image of $\frac{d}{dt}$. They furthermore satisfy $\|f - f_n\| \to 0$. Thus the image is not closed.

Furthermore, elliptic operators enjoy two further important properties, regarding the regularity of their kernel, and the dimension of that kernel.

Theorem A.23 (Elliptic regularity, [15] Thm. 4.9) *Every weak solution of an elliptic differential equation on a compact manifold is a strong solution, and in fact smooth.*

Theorem A.24 (Atiyah-Singer Index theorem, 3.7.1 in [16]) *The index of an elliptic differential operator on a compact manifold can be computed as the integral of a characteristic class. This characteristic class only depends on the symbol.*

A.6 Spectral flow

We have seen that the dimension of the moduli space of instantons on an adapted bundle can be computed in terms of the Fredholm index of the deformation operator $D_\varpi : \Omega^1(M, \operatorname{ad} P) \to \Omega^0(M, \operatorname{ad} P) \oplus \Omega^2_+(M, \operatorname{ad} P)$. In this general case, computations usually involve an index theorem, such as the Atyiah-Singer index theorem, and proceeds by cutting and gluing. However in the case of a tube $\mathbb{R} \times N$, a different approach is possible, based on the interpretation of instantons as gradient flow lines of the Chern-Simons functional. In this case, the deformation operator can equivalently seen as a Fredholm operator

$$D_\varpi : W^{1,2}(\mathbb{R}, H) \cap L^2(\mathbb{R}, W) \to L^2(\mathbb{R}, H), \quad \xi \mapsto \frac{d}{dt}\xi + L_{\omega(t)}\xi$$

where $W = W^{1,2}(\Omega^{0,1}(N, \operatorname{ad} Q))$ and $H = L^2(\Omega^{0,1}(N, \operatorname{ad} Q))$, with $W \hookrightarrow H$ being a compact inclusion of Hilbert spaces, and $L_{\omega(t)} : W \to H$ is a smooth family of self-adjoint elliptic operators on the compact manifold N. In this setting the Fredholm index of D_ϖ can be determined geometrically from the evolution of the spectrum of $L_{\omega(t)}$ via the so called **spectral flow** of a family of self-adjoint operators.

Assume more generally that W and H be two real, separable Hilbert spaces with a compact inclusion $W \hookrightarrow H$. Consider a C^1-family $A : \mathbb{R} \to \mathscr{S}(W, H)$ which converges at infinity to $A_\pm \in \mathrm{GL}(W, H)$ (in the norm topology). Under suitable conditions on A the spectra of these operators

$$S = \left\{(t, \lambda), \lambda \in \sigma(A(t))\right\} \subset \mathbb{R}^2$$

are the image of graphs of countably many C^1 functions. When such a graph intersects with the x-axis l_x at time $t_0 \in \mathbb{R}$, we call t_0 a **crossing**. More abstractly, a crossing is a time $t_0 \in \mathbb{R}$ such that the operator $A(t_0)$ has non-trivial kernel and thus is not bijective. The set of all crossings of a family A we denote by $\operatorname{cross}(A)$. In a sufficiently generic situation, all intersections of S with l_x will be transversal, and we can thus compute the intersection number $\operatorname{sf}(A) := S \cdot l_s$, which is the spectral flow of the family of operators A. Less formally, each time an eigenvalue crosses the x-axis upwards we count it as a $+1$, and each time an eigenvalue crosses downwards we count it as a -1. We will give the proper definition a bit later, but let us for now look at a few examples.

Example A.25 *Consider $W = H = \mathbb{R}$ and let $A(t)[v] = \arctan(t)v$. The spectrum of $A(t)$ is exactly $\{\arctan(t)\}$, which increases from $-\frac{\pi}{2}$ to $\frac{\pi}{2}$ and crosses the x-axis exactly once, at $t = 0$. We have $\arctan'(0) = 1 > 0$, so the spectral flow of A as described above is $+1$.*

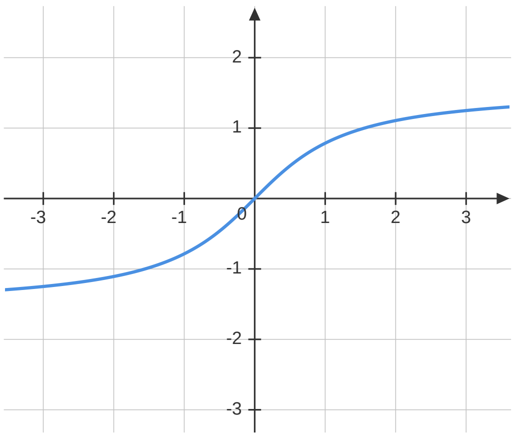

Figure A.1: Finite-dimensional spectral flow

Example A.26 *Consider $W = W^{1,2}(S^1, \mathbb{C})$ and $H = L^2(S^1, \mathbb{C})$. Let $A(t) = -i\frac{d}{ds} + \lambda(t)\,\mathrm{id}$ with $\lambda : \mathbb{R} \to \mathbb{R}$ a smooth function that converges at $\pm\infty$ to $\lambda(\pm\infty) = \pm\frac{3}{2}$. The spectrum of $A(t)$ is given by $\mathbb{Z} + \lambda(t)$, where the $(n + \lambda(t))$-eigenspace is spanned by e^{ins}. It can be checked that no matter the actual function λ, the spectral flow is given by 3, with crossings exactly when $\lambda(t) \in \mathbb{Z}$. This already indicates a certain invariance of the spectral flow under continuous change of the family while keeping the limit operators fixed.*

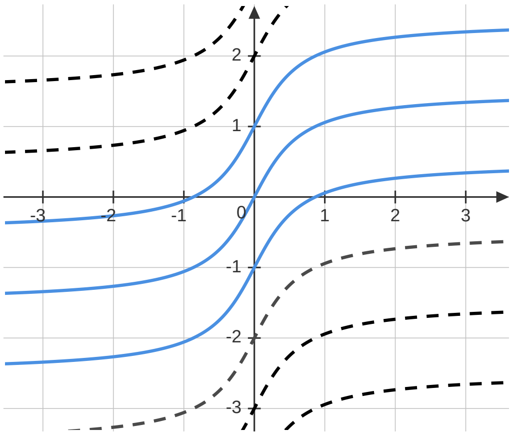

Figure A.2: Spectrum of $A(t)$ on S^1, with $\lambda(t) = \frac{3}{\pi}\arctan(2t)$.

In both of these examples it can be checked manually that the operator

$$D_A : W^{1,2}(\mathbb{R}, H) \cap L^2(\mathbb{R}, W) \to L^2(\mathbb{R}, H); \quad Du(t) = \frac{d}{dt}u(t) - A(t)u(t)$$

is injective and has a co-kernel of dimension $\mathrm{sf}(A)$, meaning that $\mathrm{ind}\, D_A = -\,\mathrm{sf}(A)$. The motivation for this correspondence between spectral flow and index comes from finite-dimensional Morse theory, wherein $W = H = \mathbb{R}^n$, and $\mathrm{ind}\, D_A$ is the dimensional of the moduli space of gradient flow lines between two critical points. In this case the requirements on A can be reformulated in saying that $A(t)$ be a symmetric matrix with $A(\pm\infty)$ having no zero eigenvalues. The index of the deformation operator is then given explicitly by the following result.

Proposition A.27 *Let $W = H = \mathbb{R}^n$ and consider $A \in \mathscr{A}(\mathbb{R}, \mathbb{R}^n, \mathbb{R}^n)$. Then:*

$$\mathrm{ind}\, D_A = \mu(A(+\infty)) - \mu(A(-\infty)),$$

*where $\mu : \mathscr{L}_{sym}(\mathbb{R}^n, \mathbb{R}^n) \to \mathbb{Z}$ is the **Morse index** of a matrix and counts its number of negative eigenvalues.*

Hence in this situation the index of D_A is precisely the change in the number of negative eigenvalues of the boundary operators. In the general setting of abstract self-adjoint operators on Hilbert spaces however, there might be an infinite number of negative and positive eigenvalues, making this definition of a naive index difficult. The spectral flow can thus be seen as a **relative index**. In a second step one can then hope to define an ad-hoc index using specific properties of the situation at hand, which for instance will lead us to a $\mathbb{Z}_8$-valued index for $\mathrm{SU}(2)$-instanton Floer homology.

We will now give the formal description of the spectral flow, following mainly [17]. Let $C = \bar{U} \subset \bar{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$ be the closure of an open set. Denote by $\mathscr{A} = \mathscr{A}(U) = \mathscr{A}(U,W,H)$ the set of continuous maps $A : U \to \mathscr{S}(W,H)$ such that $A(t)$ is invertible for each $t \in \partial U$. For instance if $U = (-\infty, 0)$ this implies that $A(0)$ as well as $A(-\infty)$ are invertible. Denote by $\mathscr{A}^1(U,W,H)$ the subset of C^1-curves, with the additional property that $\dot{A}(t) = 0$ for $t \in \partial U$. This allows to extends an element $A \in \mathscr{A}^1((a,b),W,H)$ to a C^1 curve on all of $\mathbb{R}$ by setting:

$$\tilde{A}(t) = \begin{cases} A(a), & t \leq a \\ A(t), & a < t < b \\ A(b), & b \leq t \end{cases}$$

With the notation set, we now give an axiomatic characterisation of the spectral flow, which we will afterwards see is satisfied by our intuitive definition via intersection numbers as well as by the Fredholm index of the associated operator D_A.

Theorem A.28 (Spectral flow) *For every compact inclusion $W \to H$ of separable Hilbert spaces, there exists a unique map $\mathrm{sf} : \mathscr{A}(\mathbb{R},W,H) \to \mathbb{Z}$, such that the following axioms hold:*

- *(Homotopy) The integer* sf *is constant on connected components of* $\mathscr{A}(\mathbb{R},W,H)$.
- *(Constant) If A is a constant path then* $\mathrm{sf}(A) = 0$.
- *(Direct sum) For two paths $A_i \in \mathscr{A}(\mathbb{R},W_i,H_i)$ there holds* $\mathrm{sf}(A_1 \oplus A_2) = \mathrm{sf}(A_1) + \mathrm{sf}(A_2)$
- *(Normalization) If $W = H = \mathbb{R}$ and $A(t) = \arctan(t)$ then* $\mathrm{sf}(A) = 1$.

Remark A.29 *The first axiom is to be understood as stating that the spectral flow is invariant under homotopy of the family A if the limit operators $A(\pm\infty)$ stay bijective along the whole homotopy.*

Remark A.30 *In order to define a candidate for the spectral flow* $\mathrm{sf} : \mathscr{A}(\mathbb{R},W,H) \to \mathbb{Z}$*, it is sufficient to define it on the subset $\mathscr{A}^1(\mathbb{R},W,H)$ and require it to be invariant along smooth homotopies that keep the limit operators non-singular. First, because $\mathscr{A}^1 \subset \mathscr{A}$ is dense we can extend the definition of* sf *to continuous maps. This is well-defined, as any two C^1-curves that are close enough together can be shown to be smoothly homotopic and thus have the same spectral flow. For this note that $\mathscr{S}(W,H) \subset \mathscr{L}_{sym}(W,H)$ is open dense and that* $\mathrm{GL}(W,H)$ *is open. Finally, a* *similar argument leads to the conclusion that continuous homotopies can be approximated by smooth homotopies as well, yielding a continuous map* $\mathrm{sf} : \mathscr{A}(\mathbb{R}, W, H) \to \mathbb{Z}$.

As the spectral flow is uniquely determined, providing two expressions that satisfy all the axioms has the immediate corollary that the two expressions must agree. First we express the spectral flow of A via the Fredholm index of D_A.

Let thus $A \in \mathscr{A}^1(\bar{I}, W, H)$ be a differentiable curve of self-adjoint operators, and consider the following Sobolev spaces on the interval I:

$$\mathscr{H}(I) = L^2(I, H)$$
$$\mathscr{W}(I) = L^2(I, W) \cap W^{1,2}(I, H)$$

In other words, $\mathscr{H}(I)$ consists of all measurable maps $u : I \to H$ such that

$$\|u\|^2_{\mathscr{H}} = \int_I \|u(t)\|^2_H dt < +\infty,$$

and $\mathscr{H}(I)$ consists of all absolutely continuous maps $u :\to H$ such that:

$$\|u\|^2_{\mathscr{W}} = \int_I \left(\|u(t)\|^2_W + \|\dot{u}(t)\|^2_H\right) dt < +\infty$$

We stress the fact that the weak derivative is taken with respect to the inner product $\langle \cdot, \cdot \rangle_H$. Even though u takes values in W (and thus $\dot{u}(t) \in W$ a.e.) the derivative is not necessarily in $L^2(I, W)$. Define then the operator:

$$D_A : \mathscr{W}(I) \to \mathscr{H}(I), \quad D_A u(t) := \frac{d}{dt} u(t) - A(t)u(t)$$

This is the general form of the tubular deformation operator.

Proposition A.31 (3.12 in [17]) *If* $A \in \mathscr{A}^1(\mathbb{R}, W, H)$, *then* D_A *is a Fredholm operator.*

Here it is important that the limit operators are invertible, and that for $|T| \gg 0$, the family approaches the limit operators arbitrarily well. In fact the proof that D_A has closed range relies on showing that $\|u\|_{\mathscr{W}} \leq c\,(\|D_A u\|_{\mathscr{H}} + \|Ku\|_X)$, for $K : \mathscr{H} \to X$ a compact operator. In the case of a Fredholm theory over compact manifolds, these K and X given by the Rellich compactness lemma for the inclusion $\iota : \mathscr{W} \to \mathscr{H}$. However on the non-compact domain $\mathbb{R}$, we need to replace $\mathscr{H}$ by the truncated $\mathscr{H}(T)$ to assure compactness, and K becomes the restriction $\mathscr{W} \to \mathscr{H}(T)$. If we want to prove the above inequality, it needs to hold in particular for elements $u \in \ker D_A$, i.e. they need to satisfy the bound $\|u\|_{\mathscr{W}} \leq c\|u\|_{\mathscr{H}(T)}$ for a fixed constant $c > 0$. This can only happen if the solutions decay sufficiently fast, and if for big $|T|$ the solution are close to solutions of $\frac{d}{dt}u = A(\pm\infty)u$, they will have decay of order $O(e^{-|\lambda|t})$, where $|\lambda|$ is the smallest absolute value of an eigenvalue of $A(\pm\infty)$. So if

$A(\pm\infty)$ are invertible, the decay will be exponential, which is fast enough. Although this heuristic argument is by no means a proof, it motivates why invertibility at the boundary is so important. The full proof can be found on page 9 of the paper by Robbin and Salamon [17].

Proposition A.32 *For $A \in \mathscr{A}^1(\mathbb{R}, W, H)$, we have that the assignment*

$$\mathrm{sf}_1(A) = -\operatorname{ind} D_A$$

is a spectral flow.

Proof We check the axioms for C^1 curves and C^1 homotopies by remark A.30. If the curve $A \in \mathscr{A}^1$ is continuously perturbed, then the corresponding operator D_A varies continuously through Fredholm operators. Since the index is locally constant by proposition A.14, the homotopy axiom is satisfied. The constant axiom can be verified manually using an eigen-decomposition $\{\varphi_\lambda\}_{\lambda\in\sigma(A(0))}$ for $A(0)$. If a potential element of the kernel u has a component of the form $u(0) = c\varphi_\lambda + (\dots)$ with $\lambda > 0$, then it must blow up as $t \to +\infty$, thus it is not in $W^{1,2}$. The same reasoning precludes it from having negative eigenfunctions as components. Since by assumption $0 \notin \sigma(A(0))$ this means that $u(0) = 0$, thus by the uniqueness of solutions to Lipschitz ODEs $u = 0$. A similar argument applies to the co-kernel (which is the kernel of the adjoint operator $-D_{-A}$). The direct sum axiom is again clear, since the kernel of the sum is exactly the direct sum of the kernels, and the same for co-kernels. Finally, the normalization axiom follows from the motivational result in Morse theory A.27, or can be computed by hand. □

Next, we present a different variant of the spectral flow computed in terms of intersection numbers. Let $A \in \mathscr{A}^1(U, W, H)$ be a family of operators and consider a crossing $t \in \operatorname{cross}(A)$. We define the **crossing operator** $\Gamma(A, t)$ as:

$$\Gamma(A,t) : \ker A(t) \to \ker A(t); \quad \Gamma(A,t)v = P\dot{A}(t),$$

where $P : H \to \ker A(t)$ denotes the orthogonal projection (using the Hilbert structure on H). From this operator we can extract whether or not a crossing is positive, negative or if it cannot be assigned a proper local index. We say that a crossing is **simple** if $\ker A(t)$ is one-dimensional, which means that there is exactly one eigenvalue $\lambda(t)$ crossing the x-axis at the time t. A crossing is **non-degenerate** if the crossing operator $\Gamma(A, t)$ is invertible, meaning that every crossing eigenvalue crosses transversally. This can be seen from **Kato's selection theorem** (4.28 in [17]) it is possible to pick out individual eigenvalues of a family of self-adjoint operators, and in particular for a simple crossing t this implies that there is a C^1-function $\lambda : (t-\varepsilon, t+\varepsilon) \to \mathbb{R}$ that gives exactly the eigenvalue that crosses. Non-degeneracy means in this case that $\Gamma(A,t) = \dot{\lambda}(t) \neq 0$, so that the crossing is transverse, and has a well-defined sign attached to it, $\operatorname{sgn}(\Gamma(A,t))$. For a general non-degenerate crossing the number of eigenvalues crossing up minus the number of eigenvalues crossing down is given by the signature $\operatorname{sig}(\Gamma(A,t))$ of the crossing operator (see 4.27 in [17]). From this we can define the spectral flow more geometrically:

Proposition A.33 (4.27 in [17]) *For generic $A \in \mathscr{A}^1(\mathbb{R}, W, H)$, which only has non-degenerate simple crossings, we have that*

$$\mathrm{sf}_2(A) = \sum_{t \in \mathrm{cross}(A)} \operatorname{sgn} \Gamma(A, t)$$

satisfies the axiom of a spectral flow. On A with non-degenerate (but potentially non-simple) crossings , this spectral flow is given by:

$$\mathrm{sf}_2(A) = \sum_{t \in \mathrm{cross}(A)} \operatorname{sig} \Gamma(A, t)$$

This leads us now finally to the computational tool for the Fredholm index of an operator on the tube:

Corollary A.34 (Spectral flow = -Fredholm index) *For $A \in \mathscr{A}^1(\mathbb{R}, W, H)$ with only simple crossings we have:*

$$\operatorname{ind} D_A = - \sum_{t \in \mathrm{cross}(A)} \operatorname{sgn} \Gamma(A, t)$$

A.6.1 Spectral flow on half-lines

So far we have worked with operators D_A defined on the entire real line. However sometimes it is also useful to work on two half-lines separately. For simplicity we restrict to the negative half-line, but as we will see in the next section, the results below can be equally well obtained for the positive half-line or even intervals.

We continue to work with a compact dense embedding $W \hookrightarrow H$ of separable Hilbert spaces. For $A \in \mathscr{S}(W, H)$ denote by:

$$P_A^+ : H \to \mathrm{Eig}_A^+$$

the orthogonal projection onto the closure of the eigenspaces of A corresponding to strictly positive eigenvalues. Denote by $\mathbb{R}_- := (-\infty, 0]$ the negative half-line and let $\bar{\mathbb{R}}_- := \mathbb{R}_- \cup \{-\infty\}$. We introduce the shorthand $\mathscr{A}_- := \mathscr{A}(\mathbb{R}_-)$, $\mathscr{W}^- := \mathscr{W}(\mathbb{R}_-)$ and $\mathscr{H}^- := \mathscr{H}(\mathbb{R}_-)$. Let now $A \in \mathscr{A}_-$ be a family of operators on the negative half-line. We extend the operator D_A to build boundary conditions into it:

$$F_A : \mathscr{W}^- \to \mathscr{H}^- \oplus \mathrm{Eig}_{A(0)}^+, \quad F_A u := \left(D_A u, P_{A(0)}^+ u(0)\right)$$

Note that $u(0) \in H$ is well-defined, as the trace map $W^{1,2}(\mathbb{R}_-, H) \to L^2(\partial \mathbb{R}_-, H) \simeq H$ is a bounded operator. Given a curve $A \in \mathscr{A}_-^1$ it is convenient to extend it to $\tilde{A} \in \mathscr{A}^1$ by setting $\tilde{A}(t) = A(0)$ for $t > 0$.

Proposition A.35 *Let $A \in \mathscr{A}_-^1$. Then F_A is Fredholm and*

$$\boxed{\operatorname{ind} F_A = -\mathrm{sf}(\tilde{A})} \tag{A.1}$$

The method of proof of this result will be to construct explicit isomorphisms:

$$\ker F_A \simeq \ker D_{\tilde{A}} \quad \text{and} \quad \operatorname{coker} F_A \simeq \operatorname{coker} D_{\tilde{A}}$$

and then conclude by invoking the results from [17]. Before we jump into the proof however, let us first determine the adjoint of F_A, seen as an unbounded operator $F_A : \mathscr{H}^- \to \mathscr{H}^- \oplus \mathrm{Eig}^+_{A(0)}$.

Lemma A.36 *The L^2-adjoint of F_A as an unbounded operator is given by:*

$$F_A^* : D(F_A^*) \subset \mathscr{H}^- \oplus \mathrm{Eig}^+_{A(0)} \to \mathscr{H}^-, \quad (u,h) \mapsto -D_{-A}u$$

Here $D(F_A^*) = \Big\{(f,h) \in \mathscr{H}^- \oplus \mathrm{Eig}^+_{A(0)} : f \in \mathscr{W}^- \text{ and } f(0) = -h\Big\}$.

Proof It is clear that the proposed operator is a well-defined operator with domain containing $D(F_A^*)$. We first verify the adjointness, and afterwards we show that the domain is in fact maximal. Consider $(f,h) \in D(F_A^*)$ and $u \in \mathscr{W}^-$. Then:

$$\begin{aligned}
\langle F_A u, (f,h)\rangle_{\mathscr{H}\oplus H} &= \langle D_A u, f\rangle_{\mathscr{H}} + \langle P^+_{A(0)} u(0), h\rangle_H \\
&= \langle u, -D_{-A} f\rangle_{\mathscr{H}} + \langle u(0), f(0) + h\rangle_H \\
&= \langle u, -D_{-A} f\rangle_{\mathscr{H}}
\end{aligned}$$

Notice that it is important here that f be of class $W^{1,2}$, so that we can take its trace. This proves the claim of adjointness to F_A. It is left to prove that the domain cannot be augmented. A pair $(f,h) \in \mathscr{H}^- \oplus \mathrm{Eig}^+_{A(0)}$ is in the domain of F_A^* if the functional:

$$\langle F_A(\cdot), (f,v)\rangle_{\mathscr{H}\oplus H} : C_c^\infty(\mathbb{R}_-, W) \to \mathbb{R}$$

extends to a continuous functional on $\mathscr{H}^- \oplus \mathrm{Eig}^+_{A(0)}$. Suppose first that $f \in \mathscr{H}^- \setminus \mathscr{W}^-$. Then by definition the assignment $\varphi \mapsto \langle \dot{\varphi}, f\rangle_{\mathscr{H}}$ cannot be bounded on $\mathscr{H}$. In particular we can choose a sequence of test functions φ_k which satisfy $\varphi_k(0) = 0$ and have bounded L^2-norm, so that we can write:

$$\begin{aligned}
\langle F_A \varphi_k, (f,v)\rangle_{\mathscr{H}\oplus H} &= \langle \dot{\varphi}_k, f\rangle_{\mathscr{H}} + \langle \varphi_k, -Af\rangle_{\mathscr{H}} + \langle P^+_{A(0)} \varphi_k(0), h\rangle_H \\
&= \langle \dot{\varphi}_k, f\rangle_{\mathscr{H}} + O(\|\varphi_k\|_{\mathscr{H}})
\end{aligned}$$

It is then clear that the whole functional is also unbounded on $\mathscr{H}$. So we have shown that $f \in \mathscr{W}^-$. Now onto the boundary condition. By the above computation:

$$\langle F_A(\cdot), (f,v)\rangle_{\mathscr{H}\oplus H} = \langle \cdot, -D_{-A} f\rangle_{\mathscr{H}} + \langle \mathrm{tr}(\cdot), f(0) + h\rangle_H.$$

Here $\mathrm{tr} : W^{1,2}(I) \to L^2(\partial I)$ denotes the trace operator. Consider $\varphi_k \in C^\infty(\mathbb{R}_-, W)$ which are supported in $[-\frac{1}{k}, 0]$, strictly increasing and with $\varphi_k(0) = f(0) + h$. Then $\lim_{k\to\infty} \langle F_A \varphi_k, (f,v)\rangle_{\mathscr{H}\oplus H} = \|f(0) + h\|^2_H$, however $\lim_{k\to\infty} \|u\|^2_{\mathscr{H}} = 0$. Thus the functional can only be continuous if $f(0) = -h$, and we have already checked that in this case it indeed is. □

Proof (of A.35) We will show that $\ker F_A \simeq \ker D_{\tilde{A}}$ and $\operatorname{coker} F_A \simeq \operatorname{coker} D_{\tilde{A}}$, from which the Fredholm property for F_A and the identity $\operatorname{ind} F_A = \operatorname{ind} D_{\tilde{A}}$ follow readily. Using the result A.32 above we then see that indeed $\operatorname{ind} F_A$ is computed by the spectral flow of $\tilde{A}$.

The kernel: Consider an element $u \in \ker D_{\tilde{A}}$. It satisfies the differential equation $\frac{d}{dt}u(t) = \tilde{A}(t)u(t)$, which on $\mathbb{R}_-$ simply means that $\frac{d}{dt}u(t) = A(t)u(t)$, and on $\mathbb{R}_+$ translates to $\frac{d}{dt}u(t) = A(0)u(t)$. Suppose we have an eigen-decomposition $u(0) = \sum_{\lambda\in\Lambda} \varphi_\lambda$, where $A(0)\varphi_\lambda = \lambda\varphi_\lambda$. Then the evolution equation has a solution for all positive times exactly when $\Lambda \cap \mathbb{R}_+$ is finite, and the solution is in $\mathscr{W}$ exactly when $\Lambda \cap \mathbb{R}_+ = \emptyset$. This is because the solution is given explicitly as:

$$u(t) = \sum_{\lambda\in\Lambda} e^{\lambda t}\varphi_\lambda$$

If $\Lambda \cap \mathbb{R}_+$ was infinite, then it is unbounded (by discreteness), and the arbitrary high λ that appear make it unsummable for $t > 0$. If it contains at least one positive λ (which is non-zero since $A(0)$ is invertible), then the solution blows up exponentially as $t \to +\infty$. However if only negative eigenvalues appear in the decomposition, then the solution decays sufficiently fast to be in $\mathscr{W}$. Thus if $u \in \ker D_{\tilde{A}}$, then $u|_{\mathbb{R}_-}$ is also in the kernel of D_A, and $P^+_{A(0)}u(0) = 0$, since as we have seen above the eigen-decomposition of $u(0)$ can only contain negative eigenvalues. Similarly, every $v \in \ker D_A$ such that $P^+_{A(0)}v(0) = 0$ can be extended to a solution on all of $\mathbb{R}$. Thus the two kernels agree.

The cokernel:

According to proposition 1.6 in [14] we have that $\operatorname{coker} F_A \simeq (\operatorname{im} F_A)^\perp = \ker F_A^*$. Thus we need to investigate the kernel of the adjoint operator. In fact we will apply the same argument as above to prove that $\ker F_A^* \simeq \ker D_{-\tilde{A}}$. Since $-D_{-\tilde{A}}$ is the adjoint of $D_{\tilde{A}}$ we thus arrive at $\operatorname{coker} F_A \simeq \ker F_A^* \simeq \ker D_A^* \simeq \operatorname{coker} D_A$.

By assumption, an element $(f, -f(0)) \in \ker F_A^*$ solves the equation $D_{-A}f = 0$ on the negative half-line. Since $f(0) \in \mathrm{Eig}^+_{A(0)} = \mathrm{Eig}^-_{-A(0)}$ we can extend f uniquely to a bounded solution $\tilde{f} \in \ker D_{-\tilde{A}}$. Here it is important that $f(0)$ is in the negative eigenspace of $-A(0)$, so that the solution stays bounded as $t \to +\infty$. Conversely, a bounded solution $\tilde{f} \in \ker D_{-\tilde{A}}$ restricts to an element $f \in \ker D_{-A}$ with boundary data $f(0) \in \mathrm{Eig}^+_{A(0)}$, and gives a unique element $(f, -f(0)) \in (\operatorname{im} F_A)^\perp$. Thus $\ker F_A^* \simeq \ker D_{-\tilde{A}}$, which concludes the proof. □

A.6.2 Spectral flow on general intervals

As a further generalisation, we propose to investigate the spectral flow on arbitrary intervals. Let $I = [a, b], I = (-\infty, b], I = [a, \infty)$, or $I = \mathbb{R}$ be an interval. For $I = \mathbb{R}$ or $I = \mathbb{R}_-$ we will recover the theory from above. Let $A \in \mathscr{A}(I)$ be a continuous family of operators. Note that by definition of $\mathscr{A}(I)$ one can extend A

to a continuous family $\tilde{A}$ on all of $\bar{\mathbb{R}}$ with invertible limit operator $A(\pm\infty)$. Recall that the operator $D_A : \mathscr{W}(I) \to \mathscr{H}(I)$ is a well-defined differential operator, which however does not take care of boundary conditions. In the case $I = \mathbb{R}$ this was of no importance, as any function of interest decays rapidly enough at infinity for it not to matter. In the case $I = \mathbb{R}_-$ of the previous section we introduced the operator F_A to assure that elements of its kernel could be extended onto the positive real half line. For an interval $I = [a,b]$ of finite length we introduce a further boundary condition as follows:

$$F_A : \mathscr{W}(I) \to \mathrm{Eig}^-_{A(a)} \oplus \mathscr{H}(I) \oplus \mathrm{Eig}^+_{A(b)}, \quad F_A u = \left(P^-_{A(a)} u(a), D_A u, P^+_{A(b)} u(b)\right)$$

The operators F_A for semi-infinite intervals are to be defined similarly, with however just a single boundary condition, and the operator F_A for $I = \mathbb{R}$ will be simply D_A. Similarly to the previous section one can prove that:

Proposition A.37 *Let $A \in \mathscr{A}^1(I)$, with I an interval. Then F_A is Fredholm and*

$$\boxed{\operatorname{ind} F_A = -\operatorname{sf}(\tilde{A})}$$

Proof The proof of lemma A.36 can be adapted to show that F_A^* is defined on:

$$D(F_A^*) = \left\{(v,f,u) \in \mathrm{Eig}^-_{A(a)} \oplus \mathscr{H}(I) \oplus \mathrm{Eig}^+_{A(b)} : f \in \mathscr{W}(I), f(a) = -u, f(b) = v\right\}.$$

Again, self-adjointness is clear, since the domain was chosen so that boundary terms cancel out, and maximality of the domain can be seen through a family of test functions that vanish at both ends. After that the proof of A.35 goes through in the same way. □

In addition to the structure from before, we can now introduce an additional concatenation operation. For two closed intervals $I_1, I_2 \subset \mathbb{R}$, we introduce the notation $I = I_1 \sharp I_2$ to mean that I_1 and I_2 have exactly one point in common and $I = I_1 \cup I_2$. In such a situation, we can concatenate two families $A_i \in \mathscr{A}(A_i)$ which agree at the unique intersection point of $I_1 \cap I_2$ to produce $A_1 \sharp A_2 \in \mathscr{A}(I)$. We can then generalize the axioms from A.28 as follows:

Theorem A.38 *For every compact inclusion $W \to H$ of separable Hilbert spaces and closed interval $I \subset \bar{\mathbb{R}}$, there exists a unique map $\operatorname{sf}_I : \mathscr{A}(I, W, H) \to \mathbb{Z}$, such that the following axioms hold:*

- *(Homotopy) The map sf_I is continuous.*
- *(Constant) If A is a constant path then $\operatorname{sf}_I(A) = 0$.*
- *(Direct sum) For two paths $A_i \in \mathscr{A}(I, W_i, H_i)$ there holds*

$$\operatorname{sf}_I(A_1 \oplus A_2) = \operatorname{sf}_I(A_1) + \operatorname{sf}_I(A_2).$$

- *(Extension) Let $I \subset \mathbb{R}$ with $A \in \mathscr{A}(I)$ admitting an extension $\tilde{A} \in \mathscr{A}(\mathbb{R})$. Then:*

$$\mathrm{sf}_I(A) = \mathrm{sf}_{\mathbb{R}}(\tilde{A})$$

- *(Normalization) If $W = H = \mathbb{R}$ and $A(t) = \arctan(t)$ then $\mathrm{sf}_{\mathbb{R}}(A) = 1$.*

Proof First, uniqueness. Using the homotopy, constant, direct sum and normalization axiom for $I = \mathbb{R}$ it is clear by the previous result A.28 that $\mathrm{sf}_{\mathbb{R}}$ is well-defined and unique. Suppose now that $I = [a, b]$ is a finite-length interval. Let $A \in \mathscr{A}(I)$ be a family of operators with extension $\tilde{A} \in \mathscr{A}(\mathbb{R})$. Then by the extension axiom we have that:

$$\mathrm{sf}_I(A) = \mathrm{sf}_{\mathbb{R}}(\tilde{A}),$$

and thus the spectral flow is also uniquely determined on finite length sub-intervals. The proof for semi-infinite sub-intervals is essentially the same.

Next, existence. It can be checked that the spectral flow of a C^1-family defined via intersection numbers satisfies these modified axioms, since extension is trivially satisfied, since the limit operators are invertible by assumption, which precludes crossings at the limit operators. □

Notice that in the proof of uniqueness we only required the homotopy, constant, direct sum and normalisation axioms to hold for the case $I = \mathbb{R}$. This in particular means that if we have a different candidate for spectral flow, then we only need to verify these axioms, as the other ones follow automatically from the existence of one single model for the axioms. As an immediate application we see that the spectral flow via Fredholm index is also a model for these new axioms by proposition A.37. Thus we have proven:

Theorem A.39 *For $A \in \mathscr{A}^1(I, W, H)$ with only simple crossings we have:*

$$\operatorname{ind} F_A = -\sum_{t \in \mathrm{cross}(A)} \operatorname{sgn} \Gamma(A, t)$$

We conclude this section by proving a concatenation result for $\operatorname{ind} A$ using the spectral flow via intersection numbers.

Proposition A.40 *For $I = I_1 \sharp I_2$, $A_i \in \mathscr{A}(I_i, W, H)$ and $A = A_1 \sharp A_2$ we have*

$$\mathrm{sf}_I(A) = \mathrm{sf}_{I_1}(A_1) + \mathrm{sf}_{I_2}(A_2).$$

Proof Perturb the curve A to a curve $\bar{A} \in \mathscr{A}^1(I)$ which only has simple crossings, while keeping the boundary operators non-singular. Then:

$$\begin{aligned}\mathrm{sf}_I(A) = \mathrm{sf}_I(\bar{A}) &= -\sum_{t\in\mathrm{cross}(\bar{A})} \operatorname{sgn}\Gamma(\bar{A},t) \\ &= -\sum_{t\in\mathrm{cross}(\bar{A}_1)} \operatorname{sgn}\Gamma(\bar{A}_1,t) - \sum_{t\in\mathrm{cross}(\bar{A}_2)} \operatorname{sgn}\Gamma(\bar{A}_2,t) \\ &= \mathrm{sf}_{I_1}(\bar{A}_1) + \mathrm{sf}_{I_2}(\bar{A}_2) \\ &= \mathrm{sf}_{I_1}(A_1) + \mathrm{sf}_{I_2}(A_2).\end{aligned}$$

□

Corollary A.41 *For $I = I_1 \sharp I_2$, $A_i \in \mathscr{A}(I_i, W, H)$ and $A = A_1 \sharp A_2$ we have*

$$\operatorname{ind} F_A = \operatorname{ind} F_{A_1} + \operatorname{ind} F_{A_2}.$$

Hence the indices of compatible operators are related, however note that the same is not true for the kernel and the cokernel seperately. In fact there cannot be a simple formula relating the kernel of the concatenated operator to the kernels of its constituting parts. Consider for instance $W = H = \mathbb{R}$ and an operator with $A(t) = 1$ for $|t| \geq 1$, but $A(0) < 0$. Then $\dim\ker F_{A^-} = 1$, $\dim\ker F_{A^+} = 0$ and $\dim\ker F_A = 0$. However for $A(t) = -\arctan(t+5)$, we have the same dimensions for the constituent parts, but $\dim\ker F_A = 1$.

A.6.3 Spectral flow with degenerate ends

So far we where concerned with the case where the $A \in \mathscr{A}(U)$ was invertible at every boundary point $t \in \partial U$. This was to ensure that the operator F_A is Fredholm (for instance $\frac{d}{dt} : W^{1,2}(\mathbb{R}) \to L^2(\mathbb{R})$, which corresponds to $W = H = \mathbb{R}$ and $A = 0$ is not Fredholm) and that there is well-defined sign for each crossing (the spectral flow is invariant under perturbation, but an arbitrarily small perturbation over a degenerate end can create either additional positive or negative crossings, making the spectral flow ill-defined). However in practice we sometimes do encounter families which have degenerate ends. For instance, given a reducible connection $\omega \in \mathrm{Crit}(\mathrm{CS})$ (like a trivial connection), the operator L_ω has a kernel. We would still like to associate a spectral flow to such families. To do this we have to fix a convention of whether crossings at the boundary should be counted positively or negatively. To this end, fix any smooth map $\psi : [-1,1] \to \mathbb{R}$ with $\psi(\pm 1) \neq 0$ which is constant in a neighbourhood of $\{-1, 1\}$. By translating and scaling ψ we can canonically assign a map with analogous properties to any finite interval. Similarly, by applying a fixed diffeomorphism $(-1,1) \to \mathbb{R}_{(\pm)}$ we can assign corresponding functions to (semi-) infinite case. The spectral flow we define will be dependent on the boundary data of this function. Define $\mathscr{D}(I, W, H)$ as an extension of $\mathscr{A}(I, W, H)$ where we lift the restriction that the limit operators are bijective. Then define for $A \in \mathscr{D}(I, W, H)$:

$$\mathrm{sf}'_\psi(A) = \lim_{\varepsilon\to 0^+} \mathrm{sf}(A + \varepsilon\psi\,\mathrm{id}).$$

Since by assumption the limit operators are self-adjoint, their spectra are discrete. Let $\delta > 0$ denote the minimal absolute value of an non-zero eigenvalue of the two boundary operators. The curves $A_\varepsilon = A + \varepsilon\psi\,\mathrm{id}$ for $0 < \varepsilon < \delta$ are then all in $\mathscr{A}(I, W, H)$. To see this, consider the positive limit operator A^+. It can be checked that the elements of its kernel are now in the (δ)-eigenspace of A^+_ε, and since $\delta < |\lambda|$ for every non-zero eigenvalue of A^+, no other eigenspace has been moved enough to become the kernel of the new operator. Hence the family A^+_ε is non-singular if ε is perturbed within the range $(0, \delta)$, and as such the spectral flow must be preserved. This means that the limit defining sf'_ψ is constant for small enough ε, making our new spectral flow well-defined. When the ends are non-degenerate, we can even set $\varepsilon = 0$ and still obtain a non-singular family, which implies that the new definition of spectral flow is in fact an extension of the old one, for any function ψ.

We have said above that the spectral flow sf_ψ depends on the function ψ. However it only depends on the sign of $\psi(\pm 1)$. If the homotopy class of $\psi|_{\{0,1\}} \to \mathbb{R} \setminus \{0\}$ is unchanged, then $A + \varepsilon\psi\,\mathrm{id}$ are all homotopic with non-degenerate ends for small enough epsilon, and thus thave the same spectral flow. Let us first give an account of the properties of sf_ψ before discussing the change in spectral flow when $\psi(\pm 1)$ is homotoped through 0.

Theorem A.42 *For every compact inclusion $W \to H$ of separable Hilbert spaces and closed interval $I \subset \bar{\mathbb{R}}$, there exists a unique map $\mathrm{sf}'_I : \mathscr{D}(I, W, H) \to \mathbb{Z}$, such that the following axioms hold:*

- *(Homotopy) The map sf'_I is continuous on $\mathscr{A}(I, W, H)$. Furthermore, if $A \in \mathscr{D}(\mathbb{R})$ and $A(\pm\infty)$ have no eigenvalues except potential zero eigenvalues in the range $(-\delta, \delta)$ for some $\delta > 0$, then for any $0 \leq \varepsilon < \delta$:*

$$\mathrm{sf}'_{\mathbb{R}}(A) = \mathrm{sf}'_{\mathbb{R}}(A + \varepsilon\,\mathrm{id}).$$

- *(Constant) If A is a constant path then $\mathrm{sf}'_I(A) = 0$.*
- *(Direct sum) For two paths $A_i \in \mathscr{D}(I, W_i, H_i)$ there holds*

$$\mathrm{sf}'_I(A_1 \oplus A_2) = \mathrm{sf}'_I(A_1) + \mathrm{sf}'_I(A_2).$$

- *(Extension) Let $I \subset \mathbb{R}$ with $A \in \mathscr{D}(I)$ admitting an extension $\tilde{A} \in \mathscr{D}(\mathbb{R})$. Then:*

$$\mathrm{sf}'_I(A) = \mathrm{sf}'_{\mathbb{R}}(\tilde{A})$$

- *(Normalization) If $W = H = \mathbb{R}$ and $A(t) = \arctan(t)$ then $\mathrm{sf}'_{\mathbb{R}}(A) = 1$.*

Proof First, existence. We claim that the spectral flow sf'_ψ defined above, where we choose $\psi(t) = 1$, provides an example which satisfies all of the axioms above. Since $\psi(\pm 1) > 0$ means that the kernel on both ends get pushed up, denote this spectral flow by sf_+. Let us check the axioms. Normalization is trivial, since sf_+

agrees with the usual spectral flow on non-degenerate families. Constant is also clear. Direct sum and Extension follow by applying the corresponding axioms to the perturbed families (and for Direct sum we additionally need to choose the same ε for both families). Furthermore, the first part of Homotopy also follows from the corresponding axiom for the usual spectral flow. Now if δ and ε are as above, then for ε' such that $\varepsilon + \varepsilon' < \delta$:

$$\mathrm{sf}_+(A) = \mathrm{sf}(A + \varepsilon'\psi\,\mathrm{id}) = \mathrm{sf}(A + (\varepsilon + \varepsilon')\psi\,\mathrm{id}) = \mathrm{sf}_+(A + \varepsilon\,\mathrm{id}).$$

Now onto uniqueness. Since for $A \in \mathscr{A}(I, W, H)$, the axioms reduce to the ones described in theorem A.38 for non-degenerate ends, we immediately have that $\mathrm{sf}'|_{\mathscr{A}}$ is well-defined and unique (since we already know it exists). Now suppose we have a general element $A \in \mathscr{D}(I, W, H)$. Then using extension and homotopy we have for sufficiently small ε:

$$\mathrm{sf}'_I(A) = \mathrm{sf}'_{\mathbb{R}}(\tilde{A}) = \mathrm{sf}'_{\mathbb{R}}(\tilde{A} + \varepsilon\,\mathrm{id}).$$

Now since $\tilde{A} + \varepsilon\,\mathrm{id}$ has non-degenerate ends, a property which is maintained if ε is varied in a small enough neighborhood around zero, we have determined $\mathrm{sf}'_I(A)$ uniquely from the spectral flow on non-degenerate ends, thus the entire spectral flow on degenerate ends must be unique. □

Corollary A.43 *For $I = I_1 \sharp I_2$, $A_i \in \mathscr{D}(I_i, W, H)$ and $A = A_1 \sharp A_2$ we have*

$$\mathrm{sf}'_I(A) = \mathrm{sf}'_{I_1}(A_1) + \mathrm{sf}'_{I_2}(A_2).$$

Proof By the previous theorem, we only have to verify the claim if $\mathrm{sf}' = \mathrm{sf}_+$. But there it follows directly from the corresponding result A.40 for the non-degenerate case applied to the families $A_{(i)} + \varepsilon\psi\,\mathrm{id}$, where we choose the same ε for all three families. The fact that $\psi(1) = \psi(-1)$ assures that the concatenation of the families is well-defined, and that every crossing introduced from A_1 at degenerate end $I_1 \cap I_2$ is compensated by a crossing with different sign of A_2. □

Our choice for the boundary data of ψ was to some extend arbitrary. Had we chosen $\psi(\pm 1) = -1$ instead we could have proven the analogue of theorem A.42 by replacing in the homotopy axiom $A + \varepsilon\,\mathrm{id}$ by $A - \varepsilon\,\mathrm{id}$. The problem with degenerate ends is that it is not clear if the crossings at the boundary should be counted as ending up in the positive half-space, or the negative one. By choosing $\psi(\pm 1) > 0$ we decided that the crossings should be counted as ending up negatively. A choice of $\psi(\pm 1) < 0$ would have pushed them up instead. The difference between these two approaches lies exactly in the kernel of the boundary operators. To make this more precise, denote by sf_{+-} the spectral flow corresponding to $\psi(-1) = 1$ and $\psi(+1) = -1$. We then have the following relation between sf_{+-} and sf_+.

Proposition A.44 *For $I = (a, b)$ some finite interval we have:*

$$\mathrm{sf}_{+-}(A) - \mathrm{sf}_+(A) = -\dim\ker A(b)$$

Proof Consider $\bar{A} = A\sharp U$, where $U = A(b) + \varepsilon(t-b)\,\mathrm{id}$ is defined on $(b, b+1)$ with an ε chosen so that $U(b+1)$ is non-degenerate, and there are no crossing in $(b, b+1)$. Then:

$$\mathrm{sf}_{+-}(A) = \mathrm{sf}(\bar{A}) = \mathrm{sf}_{+}(\bar{A}) = \mathrm{sf}_{+}(A) + \mathrm{sf}_{+}(U).$$

Now $\mathrm{sf}_{+}(U) = \mathrm{sf}(A(b) + (\varepsilon)(t-b)\,\mathrm{id} + \varepsilon'\,\mathrm{id})$ for small enough $\varepsilon' > 0$. There is only a single crossing of this flow (which is also non-degenerate), namely at time $t = 1 + \frac{\varepsilon'}{\varepsilon}$, where the crossing operator has kernel given by $\ker A(b)$, and thus $\mathrm{sf}_{+}(U) = -\dim\ker A(b)$, completing the proof. □

A.7 Hodge Theory

Proposition A.45 *Let $E \to (M, g)$ be a vector bundle over a Riemannian manifold. Let $\varpi \in \mathscr{C}(E)$ be a connection. Then the **Hodge Laplacian** $\Delta_\varpi = d_\varpi\delta_\varpi + \delta_\varpi d_\varpi$, which maps $\Omega^k(M, E)$ to itself is self-adjoint and elliptic.*

Proof Let $d_\varpi : \Omega^k(M, E) \to \Omega^{k+1}(M, E)$ be the exterior derivative associated to ϖ, and $\delta_\varpi : \Omega^{k+1}(M, E) \to \Omega^k(M, E)$ be its adjoint. As a connection locally has the form of a de Rham differential, its symbol is given on $\xi \in T_p^*M$ by:

$$\sigma_{d_\varpi}(\xi) = (\xi \wedge \cdot) \otimes \mathrm{id}_E$$

Thus we have for its adjoint:

$$\sigma_{\delta_\varpi}(\xi) = \sigma^*_{d_\varpi}(\xi) = (\iota_{\xi^\sharp}\cdot) \otimes \mathrm{id}_E$$

Taken together we obtain:

$$\sigma(\Delta_\varpi)(\xi) = (\xi \wedge \iota_{\xi^\sharp} \cdot + \iota_{\xi^\sharp}\xi \wedge \cdot) \otimes \mathrm{id}_E = \mathrm{id}_{\bigwedge^k T^*M} \otimes \mathrm{id}_E,$$

which is clearly an isomorphism, making Δ_ϖ elliptic.

Self-adjointness is easily verified as well, as $d^*_\varpi = \delta_\varpi$, so that:

$$\Delta^*_\varpi = (d_\varpi\delta_\varpi + \delta_\varpi d_\varpi)^* = \delta_\varpi d_\varpi + d_\varpi\delta_\varpi = \Delta_\varpi.$$ □

Now by the results from section A.3 $\Delta_\varpi|_{\Omega^k}$ is automatically Fredholm on a compact base manifold, hence has finite-dimensional kernel. It can be checked that this kernel is in fact given by $\ker\Delta_\varpi = \ker d_\varpi \cap \ker\delta_\varpi$. This has a number of applications to topology, which we will now investigate. Various other relations can be proven using the self-adjointness of Δ_ϖ and the fact that δ_ϖ is the adjoint of d_ϖ.

Theorem A.46 *Let $k \in \mathbb{N}$. Denote by Δ_ϖ the Hodge Laplacian restricted to k-forms. Then the following are orthogonal decompositions:*

- $\ker\Delta_\varpi \oplus \operatorname{im}\Delta_\varpi = \Omega^k(M, E)$

- $\ker d_\varpi \oplus \operatorname{im} \delta_\varpi = \Omega^k(M,E)$
- $\operatorname{im} d_\varpi \oplus \ker \delta_\varpi = \Omega^k(M,E)$

If the connection ϖ is flat, we have have furthermore $d_\varpi^2 = R^\varpi \wedge \cdot = 0$, so we can consider the **twisted de Rham complex**:

$$\Omega^0(M,E) \xrightarrow{d_\varpi} \Omega^1(M,E) \xrightarrow{d_\varpi} \dots \xrightarrow{d_\varpi} \Omega^{n-1}(M,E) \xrightarrow{d_\varpi} \Omega^n(M,E) \tag{A.2}$$

We denote its cohomology groups by $\mathrm{H}^k_\varpi = \mathrm{H}^k(M, d_\varpi) = \ker d_\varpi / \operatorname{im} d_\varpi$. These cohomology groups can also be represented in term of the Hodge-Laplacian:

Theorem A.47 *Denote by $\mathscr{H}^k(M, d_\varpi) := \ker \Delta^k_\varpi$ the set of* ***harmonic forms****. We have an isomorphism:*

$$\mathscr{H}^k(M, d_\varpi) \simeq \mathrm{H}^k(M, d_\varpi).$$

Proof $\ker \Delta^k_\varpi \simeq \ker d_\varpi \cap \ker \delta_\varpi = \ker d_\varpi \cap (\operatorname{im} d_\varpi)^\perp \simeq \ker d_\varpi / \operatorname{im} d_\varpi = \mathrm{H}^k(M, d_\varpi)$.

Notice that for E a trivial line bundle and ϖ a trivial connection corresponding to the Lie derivative of functions, we recover the usual de Rham complex of a manifold. Recall furthermore that for a flat ϖ corresponds to a representation $\rho \in \hom(\pi_1(M), \mathrm{GL}(n))$. In fact these homology groups also have definitions in terms of the representation alone, the so called **group cohomology** (see for instance [11]). We finish our presentation of Hodge theory by giving a generalization of Poincaré duality to the twisted setting:

Theorem A.48 (Twisted Poincaré Duality) *There is a perfect pairing*

$$\mathrm{H}^k(M, d_\varpi) \times \mathrm{H}^{n-k}(M, d_\varpi) \to \mathbb{R}$$

and thus $\mathrm{H}^k(M, d_\varpi) \simeq \mathrm{H}^{n-k}(M, d_\varpi)$.

Proof Notice that using an auxiliary metric on E, one can define an inner product on $\Omega^k(M,E)$, denoted by $\langle\!\langle \cdot, \cdot \rangle\!\rangle$. The Hodge-star operator commutes with the Hodge-Laplacian, and thus maps harmonic forms to harmonic forms. The pairing we are interested in will then be:

$$b : \mathscr{H}^k(M, d_\varpi) \times \mathscr{H}^{n-k}(M, d_\varpi) \to \mathbb{R}; \quad (\sigma, \tau) \mapsto \langle\!\langle \sigma, \star\tau \rangle\!\rangle$$

Notice that if $\tau \neq 0$, then clearly $b(\star\tau, \tau) = \|\tau\| \neq 0$, and similarly if $\sigma \neq 0$ then $b(\sigma, \star\sigma) = \pm\|\sigma\| \neq 0$. Thus the pairing is perfect. $\square$

Appendix B

Topology

In this appendix we will introduce the machinery to understand the classification of vector and principal bundles over a given topological space. We will then see applications to the case of principal $\mathrm{SU}(2)$-bundles over 3- and 4-manifolds. Next, the relation between the topology and curvature of a bundle given by the Chern-Weil homomorphism will be explained, and again the case of $\mathrm{SU}(2)$-bundles is examined in more detail. Finally, we introduce low-dimensional cobordism groups and motivate the fact that every compact oriented 3-manifold bounds a compact oriented 4-manifold.

B.1 Topology of vector bundles

Let $\mathbb{K}$ be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. Denote by $\mathrm{Vect}_{\mathbb{K}} : \mathbf{Top} \to \mathbf{Set}$ the functor which assigns to every topological space X the set of isomorphism classes of $\mathbb{K}$-vector bundles on X. If $f : Y \to X$ is a continuous map, then $\mathrm{Vect}_{\mathbb{K}}(f)$ is given by the pullback $f^* : \mathrm{Vect}_{\mathbb{K}}(X) \to \mathrm{Vect}_{\mathbb{K}}(Y)$ of vector bundles, hence it is a contravariant functor. Let $\mathrm{Vect}^n_{\mathbb{K}}$ be the functor that only assigns isomorphism classes of n-dimensional vector bundles to a base space.

Theorem B.1 (Classification via homotopy) *Let $n \leq m \in \mathbb{N}$. Let $\mathrm{Gr}_{\mathbb{K}}(m,n)$ be the Grassmannian of n-planes in $\mathbb{K}^m$. Denote by $\mathrm{Gr}_{\mathbb{K}}(\infty,n) = \varinjlim_m \mathrm{Gr}_{\mathbb{K}}(m,n)$ the Grassmannian of n-planes in $\mathbb{K}^\infty$. Then for compact X there is an isomorphism:*

$$\mathrm{Vect}^n_{\mathbb{K}}(X) \simeq [X, \mathrm{Gr}_{\mathbb{K}}(\infty,n)],$$

where the vector bundle corresponding to a map $X \to \mathrm{Gr}_{\mathbb{K}}(\infty,n)$ is given by pullback of a ***universal n-dimensional bundle*** *$E^n_{\mathbb{K}}$ over $\mathrm{Gr}_{\mathbb{K}}(\infty,n)$. A map corresponding to a vector bundle is called a* ***classifying map*** *for the vector bundle.*

The key characteristic of $\mathrm{Gr}_{\mathbb{K}}(\infty,n)$ is that it is the quotient of a contractible space $V(n,\mathbb{K}^\infty)$ (the **Stiefel manifold** of n-dimensional orthonormal frames in $\mathbb{K}^\infty$) by the action of $\mathrm{O}(n), \mathrm{U}(n)$ and $\mathrm{Sp}(n)$ respectively. A generalization of this construction will shortly lead to the **classifying space** of a Lie group.

Example B.2 *Consider the case $n = 1$. Then $\mathrm{Gr}(m,1) = \mathbb{C}P^{m-1}$ (the quotient of the space S^{2n+1} by the action of $\mathrm{U}(1)$) and hence $\mathrm{Gr}(m,1) = \mathbb{C}P^{\infty}$, which has exactly one cell in each even dimension. This implies by cellular approximation:*

$$\mathrm{Vect}^1_{\mathbb{C}}(S^2) \simeq [S^2, \mathbb{C}P^1] \simeq [S^2, S^2] \simeq \pi_2(S^2) \simeq \mathbb{Z}.$$

The isomorphism here is given explicitly by the degree of the classifying map $S^2 \to S^2$. Hence in this case the isomorphism class of a complex line bundle is completely determined by homological information.

B.1.1 Characteristic Classes

We have seen in the previous section that vector bundles are determined up to isomorphism by their classifying map, which is homotopical information and therefore hard to work with. We thus propose a different avenue to classification, which is less powerful, but is easier to compute (since it works with homology rather than homotopy), **characteristic classes**. A general characteristic class ϑ of vector bundles is a natural transformation

$$\vartheta : \mathrm{Vect}_{\mathbb{K}} \to \mathrm{H}^q(\square, A).$$

Hence it associates to every vector bundle over a space a cohomology class of a given degree and coefficient group of that space. So in particular $\vartheta(E^n_{\mathbb{K}}) \in \mathrm{H}^q(\mathrm{Gr}_K(\infty,n), A)$. Since every vector bundle $E \to X$ over a compact base space has a classifying map $f : X \to \mathrm{Gr}_K(\infty, n)$, meaning that $E \simeq f^* E^n_{\mathbb{K}}$, and characteristic classes are natural, we must have:

$$\vartheta(E) = \vartheta(f^* E^n_{\mathbb{K}}) = f^* \vartheta(E^n_{\mathbb{K}}).$$

Hence to define a characteristic class for $\mathbb{K}$-vector bundles over compact spaces, it is sufficient to pick a cohomology class in $\mathrm{H}^q(\mathrm{Gr}_K(\infty,n), A)$ for every n. This is why characteristic classes are homological in nature. Recall the cohomology rings of the classifying spaces for vector bundles:

Theorem B.3 (Homology of classifying spaces, [18])

- $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{R}}(\infty,n), \mathbb{Z}_2) \simeq \mathbb{Z}_2[w_1, w_2, \ldots, w_n]$ *where w_i has degree i.*
- $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{R}}(\infty,n), \mathbb{Q}) \simeq \mathbb{Q}[p_1, p_2, \ldots, p_{\lfloor n/2 \rfloor}]$*, where p_i has degree $4i$.*
- $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}(\infty,n), \mathbb{Z}) \simeq \mathbb{Z}[c_1, c_2, \ldots, c_n]$*, where c_i has degree $2i$.*

Using these computations be can define the following characteristic classes:

Definition B.4

- *The **Stiefel-Whitney** characteristic classes $w_i \in \mathrm{H}^i(\square, \mathbb{Z}_2)$ correspond to the generators of $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{R}}(\infty,n), \mathbb{Z}_2)$.*

- *The **Pontrjagin** characteristic classes $p_i \in \mathrm{H}^{4i}(\square, \mathbb{Z})$ correspond to the generators of* $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{R}}(\infty, n), \mathbb{Q})$.
- *The **Chern** characteristic classes $c_i \in \mathrm{H}^{2i}(\square, \mathbb{Z})$ correspond to the generators of* $\mathrm{H}^*(\mathrm{Gr}_{\mathbb{C}}(\infty, n), \mathbb{Z})$.

However these particular examples of characteristic classes can also be characterised axiomatically, as we will explain on the example of the Chern classes. For a more detailed exposition to these ideas, see [19].

Proposition B.5 (Chern classes axiomatically) *The characteristic classes c_i : Vect $\to$ $\mathrm{H}^{2i}(\square, \mathbb{Z})$ are uniquely determined by the following properties:*

1. $c_0(E) = 1$
2. $c_i(f^*E) = f^*c_i(E)$ *for all $f : X \to Y$ continuous.*
3. $c_k(E \oplus F) = \bigoplus_{i+j=k} c_i(E) \cup c_j(F)$
4. $c_1(T\mathbb{C}P^1) \neq 0$

Note that from properties 1 and 3 we can deduce that $c_i([n]) = 0$ for $i > 0$ and $[n]$ the trivial n-dimensional bundle. The Chern classes can be employed to distinguish vector bundles. However they are in general not quite enough to classify them entirely. As an example, the k-th Chern class on a compact space X descends to morphism $c_k : K_{\mathbb{C}}(X) \to \mathrm{H}^k(X)$ on complex K-theory, but the map $\mathrm{Vect}_{\mathbb{C}}(X) \to K_{\mathbb{C}}(X)$ need not be injective. Then again, sometimes they are in fact sufficient for classification.

Proposition B.6 (Classification of low-rank vector bundles) *There is an isomorphism*

$$\mathrm{Vect}^1_{\mathbb{C}}(X) \to \mathrm{H}^2(X); \quad [E] \to c_1(V),$$

given by the first Chern class.

Proof From theorem B.1 we see that $\mathrm{Vect}^1_{\mathbb{C}}(X) \simeq [X, \mathbb{C}P^\infty]$, where to each vector bundle we assign a classifying map, i.e. $E \mapsto f_E$. Since $\mathrm{H}^*(\mathbb{C}P^\infty) = \mathbb{Z}[c_1]$, all homological information of $f \in [X, \mathbb{C}P^\infty]$ is encoded in the first Chern class. It turns out that $\mathbb{C}P^\infty$ is also a $K(2, \mathbb{Z})$, meaning that all of its homotopy groups vanish, except for the second one, which is $\mathbb{Z}$. This has as a consequence that $[X, \mathbb{C}P^\infty] \simeq \mathrm{H}^2(X, \mathbb{Z})$ for every compact space X, where is the isomorphism is given as:

$$f \mapsto f^*c_1 \in \mathrm{H}^2(X, \mathbb{Z})$$

Since by naturality of the Chern class $f_E^*c_1(E^1_{\mathbb{C}}) \simeq c_1(f_E^*E^1_{\mathbb{C}}) \simeq c_1(E)$, the isomorphism:

$$\mathrm{Vect}^1_{\mathbb{C}}(X) \to [X, \mathbb{C}P^\infty] \to \mathrm{H}^2(X, \mathbb{Z})$$

is exactly given by the first Chern class. □

Chern-Weil Theory for vector bundles

So far we have seen two approaches to characteristic classes, the homotopical and the axiomatic. It turns out there is still a different way of defining characteristic classes when the base space is a smooth manifold, which relies on the relation between curvature and topology. We present the outline of lecture 37 of the notes [4].

Let V^n be a $\mathbb{K}$-vector space where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. A **polynomial function** or simply **polynomial** of degree d on V is a map $p : V \to \mathbb{C}$ such that for some (and hence every) basis $\{e_i\}_{1\leq i\leq n}$ of V^* we have for some $\alpha_I \in \mathbb{K}$:

$$p(v) = \sum_{|I|\leq d} \alpha_I e_I(v) = \sum_{|I|\leq d} \alpha_I e_{i_1}(v)\dots e_{i_{|I|}}(v).$$

We say that a polynomial p is **homogeneous** of degree d, if additionally it is of the form $p = \sum_{|I|=d} \alpha_I e_I$, i.e. all its monomials are of top degree. We denote the set of homogeneous polynomials of degree d by $P_d(V)$, and the set of all homogeneous polynomials by $P(V) = \bigoplus_{d\geq 0} P_d(V)$. Let $p \in P_d(\mathfrak{g})$ be a polynomial on the Lie algebra of a Lie group $G \subset \mathrm{GL}(n,\mathbb{K})$. We say that p is ad-**invariant** if it satisfies:

$$p(gvg^{-1}) = p(\mathrm{ad}_g\, v) = p(v)\ \forall g \in G, \forall v \in \mathfrak{g}.$$

We denote the subset of ad-invariant polynomials by $P^G(\mathfrak{g})$, respectively $P^G_d(\mathfrak{g})$ if we want to indicate the degree. Using these ad-invariant homogeneous polynomials we will now construct characteristic classes on vector bundles. Fix a homogeneous polynomial $p = \sum_{|I|=r} \alpha_I e_I \in P_r^{\mathrm{GL}(n,\mathbb{K})}(\mathfrak{gl}(n,\mathbb{K}))$ and let $E^n \to M$ be a $\mathbb{K}-$vector bundle with a connection $\nabla \in \mathscr{C}(E)$ and associated curvature form $R^\nabla \in \Omega^2(M, \mathrm{End}\, E)$. The curvature is given by local representatives $(R^\nabla)_\alpha = \sum_i \omega_i \otimes s_i \in \Omega^2(U_\alpha, \mathfrak{gl}(n,\mathbb{K}))$, so we can define:

$$\left(p(R^\nabla)\right)_\alpha = \sum_{|I|=d} \alpha_I \omega_{i_1} \wedge \cdots \wedge \omega_{i_{|I|}} e_1(s_{i_1})\dots e_{|I|}(s_{i_{|I|}})$$

By ad-invariance these local representatives fit together to form a $2r$-form $p(R^\nabla) \in \Omega^{2r}(M)$. We have the astonishing proposition:

Proposition B.7 *The $2r$-form $p(R^\nabla)$ is closed, thus determines a class in de Rham cohomology of M, and this cohomology class is independent of the connection ∇ chosen to define $p(R^\nabla)$.*

The proof relies the Bianchi identity $d^{\mathrm{End}}R^\nabla = 0$ and on the fact that the connections form an affine space (which is therefore connected). With a little more effort one can prove the following even more impressive result:

Theorem B.8 (Chern-Weil Homomorphism) *There is a ring homomorphism associated to a vector bundle $E \to M$, called the* ***Chern-Weil homomorphism****:*

$$\mathrm{CW}_E : P^{\mathrm{GL}(n,\mathbb{K})}(\mathfrak{gl}(n,\mathbb{K})) \to \mathrm{H}^*(M,\mathbb{K}).$$

It is natural with respect to pullback, meaning that for a smooth map $\varphi : M \to N$:

$$\mathrm{CW}_{\varphi^* E} = \varphi^* \circ \mathrm{CW}_E \,.$$

Example B.9 (Chern classes) *Consider $G = \mathrm{GL}(n, \mathbb{C})$ with Lie algebra $\mathfrak{gl}(n, \mathbb{C})$. Consider the characteristic polynomial associated to a matrix $X \in \mathfrak{gl}(n, \mathbb{C})$:*

$$c(\lambda X) = \det\left(\mathrm{id} + \lambda \frac{X}{2\pi i}\right).$$

Here the normalization of X is chosen so that the classes we construct will in fact be integral. The characteristic polynomial (which is also called the ***total Chern class****) however is not itself a homogeneous polynomial on $\mathfrak{gl}(n, \mathbb{C})$ as we defined it above, since it is the sum of polynomials of different degrees:*

$$\det\left(\mathrm{id} + \lambda \frac{X}{2\pi i}\right) = \sum_{i=0}^{n} c_i(X) \lambda^i$$

It is clear that the $c_i : \mathfrak{gl}(n, \mathbb{C}) \to \mathbb{C}$ are in fact in $P_i^{\mathrm{GL}(n,\mathbb{C})}(\mathfrak{gl}(n, \mathbb{C}))$. Simply note that the LHS of the above equation is ad*-invariant, and c_i is homogeneous of degree i by definition. One can prove that $\mathrm{CW}(c_i) \in \mathrm{H}^{2i}(M, \mathbb{C})$ are in fact the Chern classes that we have already defined above. This can for instance be done by checking the axioms in B.5. Axioms* 1 *and* 4 *can be explicitly be computed to be true (by setting $\lambda = 0$ and by considering the Levi-Civita connection of the round S^2 respectively), and axioms* 2 *and* 3 *are consequences of theorem B.8.*

B.2 Topology of principal bundles

Let G be a Lie group. Denote by $\mathrm{Princ}_G : \mathbf{Top} \to \mathbf{Set}$ the functor which assigns to every topological space X the set of isomorphism classes of G-principal bundles on X. If $f : Y \to X$ is a continuous map, then $\mathrm{Princ}_G(f)$ is given by the pullback $f^* : \mathrm{Princ}_G(X) \to \mathrm{Princ}_G(Y)$ of principal bundles, hence it is a contravariant functor.

Theorem B.10 (Classification via homotopy) *Let G be a compact Lie group and X be a compact topological space. Then there is a topological space* BG *as well as a contractible space* EG*, such that* EG $\to$ BG *is a principal G-bundle, called the* ***universal bundle****. Every principal G-bundle $P \to X$ can be obtained via pullback of* EG *along a map $f : X \to \mathrm{BG}$, called its* ***classifying map****, and the homotopy class of f uniquely determines the isomorphism type of P, hence:*

$$\mathrm{Princ}_G(X) \simeq [X, \mathrm{BG}].$$

Alternatively every principal bundle has a presentation in terms of transition functions. Given an atlas $\mathfrak{U} = \{(U_\alpha, \varphi_\alpha)\}_{\alpha \in A}$ of the base manifold M, there are transition functions $g_{\alpha,\beta} : U_\alpha \cap U_\beta \to G$ such that $P \simeq \bigsqcup_{\alpha \in A} \{a\} \times U_\alpha \times G/\sim$, where for (α, p) with $p \in U_\alpha \cap U_\beta$ we have $(\alpha, p, g) \sim (\beta, \varphi_\beta \circ \varphi_\alpha^{-1}(p), g_{\alpha\beta}(p)g)$. In fact its isomorphism type is determined by this **Čech cocycle**.

Theorem B.11 (Classification via cohomology, [20] Thm 3.6)

$$\mathrm{Princ}_G(X) \simeq \check{\mathrm{H}}^1(X, C^\infty(\cdot, G)),$$

where on the right-hand-side we mean Čech cohomology with coefficients in the sheaf of G-valued functions.

In the case of G having the discrete topology, sheaf cohomology reduces to regular cohomology, hence by combining the two theorems above with the representability of cohomology by the Eilenberg-Maclane spectrum we obtain:

Corollary B.12 *Let G be a discrete group. Then* $\mathrm{Princ}_G(X) \simeq \check{\mathrm{H}}^1(X, G) \simeq \mathrm{H}^1(X, G) \simeq [X, K(G, 1)]$ *naturally and hence* $\mathrm{BG} \sim K(G, 1)$.

We are now in a position to classify all bundles which are relevant for our purposes. Let $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$. Then $\mathrm{B}\mathbb{K}^\times \simeq \mathbb{K}P^\infty$ and so in particular:

Theorem B.13 *Let X be any topological space. Then:*

$$\mathrm{Princ}_{\mathrm{SU}(2)}(X) \simeq [X, \mathbb{H}P^\infty]$$

If X is a smooth oriented manifold of dimension 4, $\mathrm{Princ}_{\mathrm{SU}(2)}(X) \simeq \mathbb{Z}$. *If X is a CW-complex of dimension less than* 3, *every principal bundle over X is trivial.*

Proof If X is a smooth oriented manifold of dimension 4, by cellular approximation:

$$\mathrm{Princ}_{\mathrm{SU}(2)}(X) \simeq \mathrm{Princ}_{\mathbb{H}^\times}(\mathrm{SU}(2)) \simeq [X, \mathbb{H}P^1] \simeq [X, S^4] \simeq \pi^4(X),$$

the fourth cohomotopy group of X. This is because $\mathrm{SU}(2)$ can be identified with the unit sphere in $\mathbb{H}$. By the Pontrjagin construction (see for instance section 7 in [6]):

$$\pi^4(X) \simeq \mathbb{Z},$$

where the isomorphism is given by the degree. If X is a CW-complex of dimension less than 3, again by cellular approximation:

$$\mathrm{Princ}_{\mathrm{SU}(2)}(X) \simeq [X, \mathbb{H}P^0] \simeq [X, \star] = 0,$$

since $\mathbb{H}P^\infty$ admits exactly one cell in dimension $4k, k \in \mathbb{N}$. □

B.2.1 Characteristic classes

As in the case for vector bundles, **characteristic classes for principal bundles** are cohomology classes in the base manifold that behave naturally with respect to morphisms of principal bundles, and all the same remarks apply. We will now present the Chern-Weil theory for unitary bundles, as it will allow us to relate the topology of a principal bundle to its curvatue.

Chern-Weil Theory

Let $P \in \mathrm{Princ}_G(M)$ be a principal G-bundle over a smooth manifold M. Similarly as before, one can associate to an ad-invariant homogeneous polynomial $p \in P_d^G(\mathfrak{g})$ a de Rham cohomology class. Again we could choose either real or complex coefficients, but since we are mostly interested in $\mathrm{SU}(2)$-bundles, we'll restrict to complex coefficients. However this time we consider the curvature as a form $\mathrm{F}_\varpi \in \Omega^2(M, \mathrm{ad}\, P)$ instead of $R^\nabla \in \Omega^2(M, \mathrm{End}\, E)$. Since the sections of ad P transform under the adjoint representation when changing charts, ad-invariance again provides well-definedness of the form $p(\mathrm{F}_\varpi)$. The relevant form of the Bianchi identity $d_\varpi \mathrm{F}_\varpi = 0$ then allows to prove the Chern-Weil theorem in the setting of principal bundles:

Theorem B.14 (Chern-Weil Homomorphism) *Let $P \in \mathrm{Princ}_G(M)$. Then there is an algebra homomorphism, called the* ***Chern-Weil homomorphism****:*

$$\mathrm{CW}_P : P^G(\mathfrak{g}) \to \mathrm{H}^*(M, \mathbb{C})$$

It is natural with respect to pullback, meaning that for a smooth map $\varphi : M \to N$:

$$\mathrm{CW}_{\varphi^* P} = \varphi^* \circ \mathrm{CW}_P .$$

Example B.15 (Chern classes) *Let $\iota : G \hookrightarrow \mathrm{GL}(n, \mathbb{C})$ be a matrix Lie group. Consider again the characteristic polynomial with its decomposition into homogeneous terms:*

$$\det\left(\mathrm{id} + \lambda \frac{\square}{2\pi i}\right) = \sum_{i=0}^{n} c_i(\square)\lambda^i : \mathfrak{g} \to \mathbb{C}.$$

These are again well-defined characteristic classes, the ***Chern classes*** *of P. As G is a matrix Lie group, every P has a distinguished associated bundle, namely*

$$E_P = P \times_\iota \mathbb{C}^n.$$

It is the (up to isomorphism) unique vector bundle that has the same transition maps $g_{\alpha\beta} : U_{\alpha\beta} \to G \subset \mathrm{GL}(n, \mathbb{C})$ as P. Their Chern classes are identical:

$$\mathrm{CW}_P(c_i) = \mathrm{CW}_{E_P}(c_i) \in \mathrm{H}^{2i}_{dR}(M, \mathbb{C}).$$

Example B.16 (Chern classes on SU-bundles) *In the case of an $\mathrm{SU}(2)$-bundle, the Chern classes are given by the formulae:*

- $c_0(\mathrm{F}_\varpi) = 1$
- $c_1(\mathrm{F}_\varpi) = 0$
- $c_2(\mathrm{F}_\varpi) = \frac{1}{8\pi^2} \mathrm{tr}(\mathrm{F}_\varpi \wedge \mathrm{F}_\varpi)$

Proof Let $X = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathfrak{su}(2)$ be given. Recall that the Chern classes are given as the Taylor expansion of the total Chern class, which for $t \in \mathbb{R}$ is :

$$c(tX) = \det\left(\mathrm{id} + \frac{tX}{2\pi i}\right) = \sum_{i=0}^{2} t^i c_i(X)$$

And we compute explicitly:

$$\det\left(\mathrm{id} + \frac{tX}{2\pi i}\right) = \det\begin{pmatrix} 1 + \frac{at}{2\pi i} & \frac{bt}{2\pi i} \\ \frac{ct}{2\pi i} & 1 + \frac{dt}{2\pi i} \end{pmatrix} = 1 + t\frac{a+d}{2\pi i} + t^2\frac{bc-ad}{4\pi^2}$$

Hence $c_0(X) = 1$ and $c_1(X) = \frac{\mathrm{tr}(X)}{2\pi i} = 0$ since $\mathfrak{su}(2)$-matrices are traceless. A further computation yields: $bc - ad = \frac{(a^2+bc+cb+d^2)-2(a+d)^2}{2} = \frac{\mathrm{tr}(X^2)-\mathrm{tr}^2(X)}{2} = \frac{\mathrm{tr}(X^2)}{2}$, which gives the formula for c_2. □

B.3 Cobordism groups

The **oriented cobordism group** Ω_n^{SO} has as underlying set the oriented manifolds of dimension n up to oriented cobordism. This means that two manifolds M and N are considered equivalent if there is an oriented $(n+1)$-manifold W such that $\partial W = M \cup \bar{N}$ with matching orientations. The group operation is the disjoint union of manifolds.

Let M be an n-dimensional oriented manifold with distinguished fundamental class $[M] \in \mathrm{H}_n(M)$. Consider a partition $n = p_1 + 2p_2 + \cdots + np_n$, meaning that $p_i \in \mathbb{N}$ for $1 \leq i \leq n$. Denote the vector of integers $(p_1, \ldots, p_n)$ by $\mathbf{p}$. We can then evaluate the cohomology class

$$w_{\mathbf{p}} = \bigcup_{i=1}^{n} (w_i(TM))^{p_i} \in \mathrm{H}^n(M)$$

on the fundamental class $[M]$ to obtain the $(p_1, \cdots, p_n)$-**Stiefel-Whitney number**:

$$\sharp w_{\mathbf{p}} = w_{\mathbf{p}}([M]) \in \mathbb{Z}$$

Similarly, we can define the **Pontrjagin numbers** and for complex manifold the **Chern numbers**. They provide a way to compare the tangent bundles of two manifolds, even though these do not have the same base space, which makes it impossible to compare cohomology classes directly. Diffeomorphic manifolds clearly have the same characteristic numbers, but non-diffeomorphic ones need not have different numbers. In fact the classification strength lies somewhere in between being able to determine diffeomorphism types and being useless. More precisely we have the following result:

Theorem B.17 (Wall, 1960, [19]) *Two oriented manifolds are oriented cobordant iff their Stiefel-Whitney and Pontrjagin numbers agree.*

From this theorem and the fact that only finitely many characteristic numbers are non-vanishing for manifolds of a given dimension (as there are only finitely many ways to partition an integer) we obtain the following corollary:

Corollary B.18 Ω_n^{SO} *is finitely generated.*

Let us apply this machinery to our case of interest, dimension three. In this dimension there are no non-zero Pontrjagin classes, w_1 vanishes because we only consider oriented manifolds, and w_2 vanishes since every oriented three-dimensional manifold is spin (Theorem 1 on page 46 of [21]), we have

$$\Omega_3^{\mathrm{SO}} = 0.$$

Hence, every three-dimensional manifold is the boundary of a compact four-dimensional manifold.

Bibliography

[1] S.K. Donaldson. *Floer Homology groups in Yang-Mills theory*. Cambridge University Press, 2002.

[2] Jonny Evans. Aspects of Yang-Mills Theory, 2011.

[3] Michael Struwe. *Variational Methods*. Springer, 2008.

[4] Will J. Merry. Lecture notes on Differential Geometry, August 2019.

[5] John W. Morgan. *An Introduction to Gauge Theory*. IAS/Park City Mathematics Series, 1998.

[6] John Milnor. *Topology from the Differentiable Viewpoint*. Princeton University Press, 1965.

[7] Andreas Floer. An instanton-invariant for 3-manifolds. *Comm. Math. Phys.*, 118(2):215–240, 1988.

[8] Serge Lang. *Differential and Riemannian Manifolds*. Springer, 1995.

[9] S.K. Donaldson and P.B. Kronheimer. *The Geometry of Four-Manifolds*. Clarendon Press, Oxford, 1990.

[10] J.W. Morgan. *The Seiberg-Witten Equations and Applications to the Topology of Smooth Four-Manifolds*. Princeton University Press, 1995.

[11] Nikolai Saveliev. *Invariants for Homology 3-Spheres*, volume 140. Springer, 2002.

[12] Michael Francis Atyiah. *K-Theory*. Addison-Wesley Publishing, 1964.

[13] Joachim Kock. *Frobenius Algebras and 2D topological Quantum Field Theories*. 2003.

[14] Schmüdgen Konrad. *Unbounded Self-adjoint Operators on Hilbert Space*. Springer, 2012.

[15] Raymond O. Jr. Well. *Differential Analysis on Complex Manifolds*. Springer, 2008.

[16] Peter B. Gilkey. *Invariance Theory, the Heat Equation, and the Atyiah-Singer Index Theorem*. CRC Press, 1995.

[17] Joel Robbin and Dietmar Salamon. The spectral flow and the maslov index. *Bulletin of the London Mathematical Society*, 27(1):1–33, 1995.

[18] Edgar H. Brown. The cohomology of bso_n and bo_n with integer coefficients. *Proceedings of the American Mathematical Society*, 85(2):283–288, 1982.

[19] James Stasheff John Milnor. *Characteristic classes*. Princeton University Press, 1974.

[20] Max Karoubi. *K-Theory, An Introduction*. Springer, 1978.

[21] Robion C. Kirby. *3-Manifolds; a digression*. Springer Berlin Heidelberg, Berlin, Heidelberg, 1989.

List of Symbols

$\mathscr{X}(M)$	vector fields on M
$\star_g = \star : \Omega^k \to \Omega^{n-k}$	Hodge star operator
$\star_3, \star_4 = \star$	Hodge star on 3-/4-manifold
$d, d_\varpi : \Omega^k \to \Omega^{k+1}$	(twisted) de Rham differential
$\delta, \delta_\varpi : \Omega^k \to \Omega^{k-1}$	adjoint of (twisted) de Rham differential
$\Delta, \Delta_\varpi : \Omega^k \to \Omega^k$	(Hodge) Laplacian
$\sharp, \flat : \Omega^k_G(P, \mathfrak{g}) \leftrightarrow \Omega^k(M, \operatorname{ad} P)$	isomorphism for equivariant forms
$\mathscr{C}(P), \mathscr{C}(E)$	connections on principal/vector bundle
$\mathscr{C}^{k,p}(P)$	connections of Sobolev class $W^{k,p}$
ω, ϖ	connection on $3-$ and $4-$manifold.
$A, \mathbf{A}$	connection potential on $3-$ and $4-$manifold.
$\nabla^\varpi : \Omega^0 \to \Omega^1$	induced connection on associated bundle
$R^\nabla \in \Omega^2(M, \operatorname{End}(E))$	curvature of connection on vector bundle
$\Omega \in \Omega^2(P, \mathfrak{g})$	curvature of connection on principal bundle
$\mathrm{F}_\varpi = \Omega^\flat \in \Omega^2(M, \operatorname{ad} P)$	curvature potential
$\mathrm{F}_\mathbf{A} = s^*\Omega \in \Omega^2(U, \mathfrak{g})$	curvature potential in trivialisation
$\mathscr{G}(P), \mathscr{G}^{k,p}(P)$	gauge group of principal bundle (of class $W^{k,p}$)
$\operatorname{Ad} P = P \times_c G$	non-linear adjoint fibre bundle
$\mathscr{B}^*(P)$	moduli space of (irreducible) connections
$\mathscr{B}^{k,p}(P), \mathscr{B}^{*,k,p}(P)$	moduli space of class $W^{k,p}$
$\mathrm{R}(P) \subset \mathscr{C}(P)$	flat connections
$\mathscr{R}^{(*)}(P) \subset \mathscr{B}^{(*)}(P)$	gauge equivalence classes of flat connections
$\mathscr{C}(M), \mathscr{B}(M), \mathscr{G}(M)$	corresponding sets for a trivial bundle over M

YM	Yang-Mills functional
$\omega^{\pm} \in \Omega^2$	(anti-)self-dual part of two-form
$\mathrm{F}^{+}(\varpi)$	self-dual component of curvature
D_{ϖ}	four-dimensional instanton deformation operator
CS	Chern-Simons functional
$\operatorname{grad}\mathrm{CS}$	gradient w.r.t. L^2-norm of CS
$\operatorname{Hess}_{\omega}\mathrm{CS}$	Hessian of CS as bilinear form
$H_{\omega} = d\operatorname{grad}\mathrm{CS}(\omega)$	Hessian as linear operator
$L_{\omega} = S_{\omega} \oplus H_{\omega}$	three-dimensional deformation operator of CS
$\lambda(N)$	Casson invariant of three-manifold
$\delta([\eta],[\rho])$	relative Floer index
$\mu([\rho])$	absolute Floer index
$\mathscr{M}_P$	moduli space of instantons on adapted bundle
τ_T	gluing map with parameter $T \in \mathbb{R}_{>0}$
$\mathrm{CF}^{\bullet}(N)$	Floer chain complex
∂_g	Floer boundary operator (dependen on metric)
$\mathrm{HF}^{\bullet}(N,g) \simeq \mathrm{HF}^{\bullet}(N)$	Floer homology groups
nCob	n-dimensional oriented cobordism category
$\Gamma(E)$	smooth sections of vector bundle
$L^p(E)$	Lebesgue class sections of vector bundle
$W^{k,p}(E)$	Sobolev class sections of vector bundle
$\sigma_P(\xi)$	symbol of partial differential operator
$\sigma(A)$	spectrum of linear operator
$\mathrm{sf}, \mathrm{sf}_I$	spectral flow
sf'_{ψ}	perturbed spectral flow
$\mathrm{H}^{\bullet}(M, d_{\varpi})$	twisted de Rham cohomology groups